Optimal control and dissipativity: Stability and efficient numerics

Manuel Schaller, Optimization-based Control, Technische Universität Ilmenau

Trends on dissipativity in systems and control, UniDistance Suisse, 25.05.2022

Outline

1. Dissipative optimal control problems
2. The exponential turnpike property
3. Port-Hamiltonian optimal control
4. Efficient dissipativity-exploiting numerics for optimal control

Part 1: Dissipative optimal control problems

Dissipative dynamical system

Consider a **dynamical system**

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0$$

with $x : \mathbb{R}^+ \to X$ and $u : \mathbb{R}^+ \to U$, X **and** U **Hilbert spaces**.

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Definition

A control system is called **dissipative** if there is a storage function $S : X \to \mathbb{R}^{\geq 0}$ and supply rate $s : X \times U \to \mathbb{R}$ such that

$$\underbrace{S(x(T)) - S(x(0))}_{\text{change in stored energy}} \leq \underbrace{\int_0^T s(x(t), u(t))\, \mathrm{d}t}_{\text{supplied energy}}$$

along trajectories.

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Strictly dissipative optimal control problem

Consider the Optimal Control Problem (OCP)

$$\min_{u \in L_\infty(0,T;U)} \int_0^T \ell(x(t),u(t))\,\mathrm{d}t \quad \text{s.t.} \quad \dot{x}(t) = f(x(t),u(t)), \quad x(0) = x_0$$

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Definition (e.g. Faulwasser '17, Grüne '22)

An OCP is **strictly dissipative at a controlled steady state** $(\bar{x},\bar{u})$ $(f(\bar{x},\bar{u}) = 0)$ if there is a storage function $S : X \to \mathbb{R}^{\geq 0}$ such that

$$S(x(T)) - S(x(0)) \leq \int_0^T \ell(x(t),u(t)) - \ell(\bar{x},\bar{u}) - \alpha(\|x(t)-\bar{x}\|)\,\mathrm{d}t$$

where

$\alpha \in \mathcal{K}_\infty = \{\phi : [0,\infty) \to [0,\infty) \,|\, \phi(0) = 0,\ \phi \text{ strictly increasing and unbounded}\}$.

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Asymptotic stability of optimal solutions: the turnpike property

Definition (Turnpike properties)

- An OCP has the **exponential turnpike property** on a set of initial values $\mathbb{X}_0 \subset X$ with respect to $\bar{x} \in X$ if there is $c, \mu > 0$ such that for all $T > 0$ and $x_0 \in \mathbb{X}_0$ we have for optimal solutions x^* that

$$\|x^*(t) - \bar{x}\| \leq c \left(e^{-\mu t} + e^{-\mu(T-t)} \right)$$

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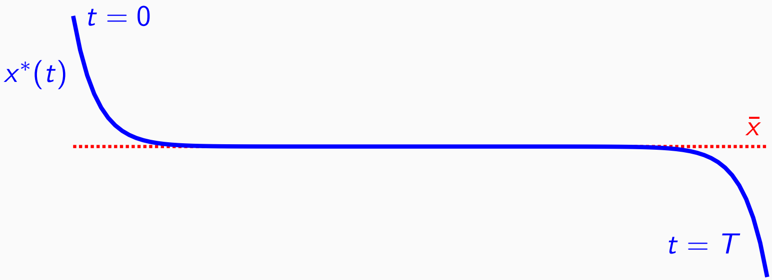

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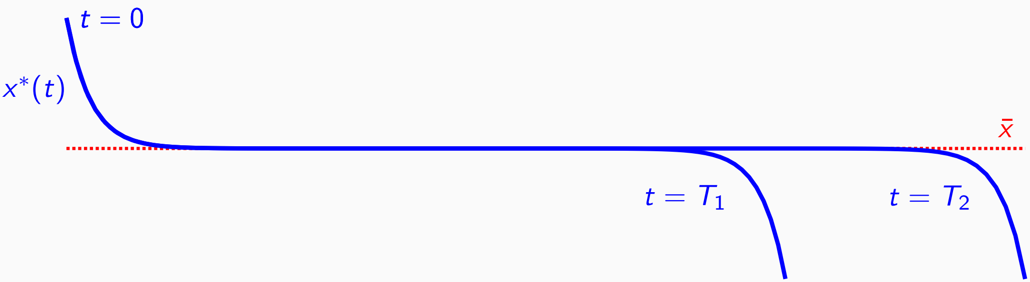

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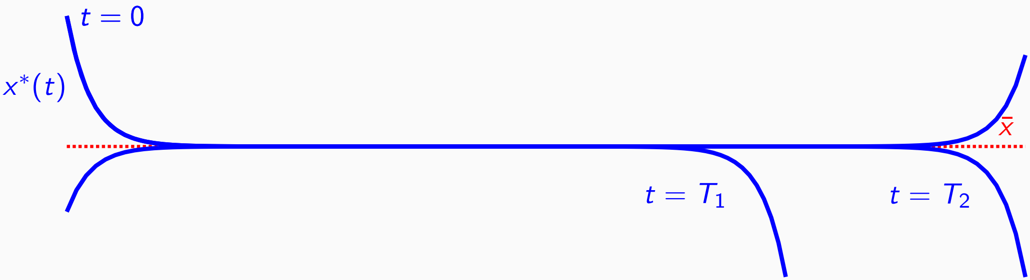

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Definition (Turnpike properties)

- An OCP has the **integral turnpike property** on a set of initial values $\mathbb{X}_0 \subset X$ with respect to $\bar{x} \in \mathbb{X}$ if **there is** $\beta \in \mathcal{K}$ **and** $c > 0$ such that for all $T > 0$,

$$\int_0^T \beta(\|x^*(t) - \bar{x}\|)\,\mathrm{d}t \leq c$$

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- An OCP has the **measure turnpike property** on a set of initial values $\mathbb{X}_0 \subset X$ with respect to $\bar{x} \in \mathbb{X}$ if **for each** $\varepsilon > 0$**, there is** C_ε such that for all $T > 0$

$$\left|\{t \in [0, T] \mid \|x^*(t) - \bar{x}\| > \varepsilon\}\right| \leq C_\varepsilon$$

Relation to dissipativity

Under controllability assumptions we have:

- str. dissipativity $\Longrightarrow$ measure/integral turnpike (Carlson et al. '91, Grüne '13, Faulwasser et al. '17, Zhang&Trelat '18)

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- turnpike $\Longrightarrow$ str. dissipativity (Grüne&Müller '16)

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Stability and suboptimality of Model Predictive Control (MPC)

If the **OCP is strict dissipative** w.r.t. $\bar{x}$ then:

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- Suboptimality estimates in different flavors, e.g.,

$$V(x_0) \leq \text{MPC cost} \leq V(x_0) + R(\text{MPC horizon})$$

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Verifying dissipativity in nonlinear (optimal) control

Some possible routes:

- Equivalence to detectability (+additional assumptions): Höger&Grüne '19

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Can be **as hard as finding a Lyapunov function.** In particular, if ℓ and f are **given**.

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Dissipativity in linear quadratic optimal control

$$\ell(x,u) = \frac{1}{2}\left(\|Cx\|^2 + \|u\|^2\right) + \langle z, x\rangle + \langle v, u\rangle,$$
$$f(x,u) = Ax + Bu$$

Damm, Grüne, Stieler, & Worthmann (2014). An exponential turnpike theorem for dissipative discrete time optimal control problems. SICON, 52(3), 1935-1957.

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- X Hilbert space: (A, C) detec. $\Rightarrow$ strict pre-dissipative (Grüne&Philipp&S. '22)

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Part 2: The exponential turnpike property

Turnpike in view of the optimality system: linear quadratic problems

Setting: $A : D(A) \subset X \to X$ generates **s.c. semigroup**, $C \in L(X, Y)$, $B \in L(U, X)$.

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\begin{aligned}
\min_u \ & \frac{1}{2} \int_0^T \|C(x(t) - x_d)\|_Y^2 + \alpha \|u(t)\|_U^2 \, dt \\
\text{s.t.} \quad & \dot{x}(t) = Ax(t) + Bu(t) + f, \\
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Pontryagin Maximum Principle

$$\begin{pmatrix} \dot{\lambda}(t) \\ \dot{x}(t) \\ 0 \end{pmatrix} = \begin{pmatrix} -A^* & -C^*C & 0 \\ 0 & A & B \\ B^* & 0 & \alpha I \end{pmatrix} \begin{pmatrix} \lambda(t) \\ x(t) \\ u(t) \end{pmatrix} + \begin{pmatrix} C^*Cx_d \\ f \\ 0 \end{pmatrix}$$

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KKT-conditions

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KKT-conditions

$$
\begin{pmatrix} \dot{\bar{\lambda}} \\ \dot{\bar{x}} \end{pmatrix} = \begin{pmatrix} -A^* & -C^*C \\ -\frac{1}{\alpha}BB^* & A \end{pmatrix} \begin{pmatrix} \bar{\lambda} \\ \bar{x} \end{pmatrix} + \begin{pmatrix} C^*Cx_d \\ f \end{pmatrix}
$$

with $\bar{x}(0) = \bar{x}$, $\bar{\lambda}(T) = \bar{\lambda}$.

Grüne, S., Schiela (2020). Exponential sensitivity and turnpike analysis for linear quadratic optimal control of general evolution equations. JDE, 268(12), 7311-7341.

Infinite-dimensional LQR: exponential turnpike

Theorem (Grüne, S., Schiela, '20)

(A, B) exp. stabilizable, (A, C) exp. detectable. Then there is $\mu, c > 0$ such that for all $T > 0$

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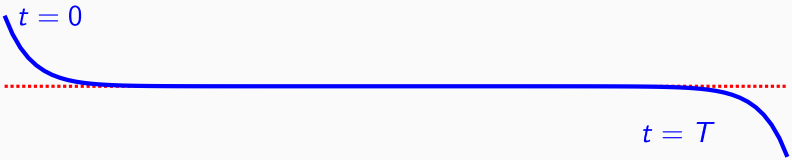

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Proof idea

Preparation: Define an operator corresponding to the optimality conditions:

$$\underbrace{\begin{pmatrix} -\frac{d}{dt}-A^* & C^*C \\ E_T & 0 \\ -BB^* & \frac{d}{dt}-A \\ 0 & E_0 \end{pmatrix}}_{=:M} \underbrace{\begin{pmatrix} \lambda(t)-\bar{\lambda} \\ x(t)-\bar{x} \end{pmatrix}}_{=:\delta z} = \underbrace{\begin{pmatrix} 0 \\ -\bar{\lambda} \\ 0 \\ x_0-\bar{x} \end{pmatrix}}_{=:\varepsilon}$$

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Step 2: Neumann series argument for $(I+\mu F M^{-1})^{-1}$ assuming $\mu\|FM^{-1}\| < 1$:

$$\|\widetilde{\delta z}\| \leq c\|M^{-1}\|\|\tilde{\varepsilon} \leq c\|M^{-1}\| \left(\|\bar{\lambda}\| + \|x_0 - \bar{x}\|\right)$$

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Step 3: Deduce bound on $\|M^{-1}\|$ uniformly in T via stabilizability and detectability.

Extensions and refinements

Extensions:

- **Admissible** input operator $B \notin L(U, X)$: case of boundary control.

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S. (2021). Sensitivity Analysis and Goal Oriented Error Estimation for Model Predictive Control, PHD thesis, University of Bayreuth

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Extensions:

- **Admissible** input operator $B \notin L(U, X)$: case of boundary control.
- **Terminal conditions** $x(T) = x_T$: exact controllability.
- **Analytic** semigroups: Turnpike in stronger norm $\|x\|_{L_2(0,T;D(A))} + \|x'\|_{L_2(0,T;X)}$

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Optimal control problem:

$$\min_{u\in L^\infty(0,T;U)} \int_0^T \ell(x(t),u(t))\mathrm{d}t$$

$$\text{s.t.} \quad \dot{x}(t) = f(x(t),u(t)), \;\; x(0) = x_0$$

Turnpike in view of the optimality system: Nonlinear case

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Pontryagin Maximum Principle

$$H(x,\lambda,u) = \ell(x,u) + \lambda^\top f(x,u)$$

$$\begin{aligned} \dot{\lambda}(t) &= -H_x(x(t),\lambda(t),u(t)) \\ \dot{x}(t) &= H_\lambda(x(t),\lambda(t),u(t) \\ 0 &= H_u(x(t),\lambda(t),u(t)) \end{aligned}$$

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KKT-conditions

$$L(\bar{x},\bar{\lambda},\bar{u}) = H(\bar{x},\bar{\lambda},\bar{u}).$$

$$\begin{aligned}0 &= -L_x(\bar{x},\bar{\lambda},\bar{u}) &&= -H_x(\bar{x},\bar{\lambda},\bar{u})\\ 0 &= L_\lambda(\bar{x},\bar{\lambda},\bar{u}) &&= H_\lambda(\bar{x},\bar{\lambda},\bar{u})\\ 0 &= L_u(\bar{x},\bar{\lambda},\bar{u}) &&= H_u(\bar{x},\bar{\lambda},\bar{u})\end{aligned}$$

Turnpike for nonlinear problems via optimality system

- **Local** TP in finite dimension (sketch): Trélat&Zuazua'15.

Trélat, Zuazua (2015). The turnpike property in finite-dimensional nonlinear optimal control. JDE, 258(1), 81-114.

Grüne, S., Schiela (2021). Abstract nonlinear sensitivity and turnpike analysis and an application to semilinear parabolic PDEs. ESAIM:COCV, 27, 56.

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Part 3: Port-Hamiltonian optimal control

Port-Hamiltonian systems

Consider $J(x) = -J(x)^\top$, $R(x) = R(x)^\top \geq 0$, the **Hamiltonian** $H : \mathbb{R}^n \to \mathbb{R}$ and the **port-Hamiltonian dynamics**

$$\begin{aligned} \frac{d}{dt}x(t) &= (J(x(t)) - R(x(t)))H_x(x(t)) + g(x(t))u(t) \\ y(t) &= g(x(t))^\top H_x(x(t)) \end{aligned}$$

Faulwasser, Flaßkamp, Ober-Blöbaum, S., Worthmann (2022). Manifold turnpikes, trims, and symmetries. MCSS, https://doi.org/10.1007/s00498-022-00321-6

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Energy balance:

$$H(x(T)) - H(x(0)) = \int_0^T \underbrace{\langle u(t), y(t)\rangle}_{\text{supplied power}} - \underbrace{\left\| R^{\frac{1}{2}}(x(t))H_x(x(t))\right\|^2}_{\text{dissipated power}} \,\mathrm{d}t$$

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Strict dissipative OCP w.r.t. steady state $\bar{x}$, ($\ell(\bar{x}, \bar{u}) = 0$):

$$S(x(T)) - S(x(0)) \leq \int_0^T \ell(x(t), u(t)) - \alpha(\|x(t) - \bar{x}\|)\,\mathrm{d}t$$

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Strict dissipative OCP w.r.t. set $\mathcal{V} \subset \mathbb{R}^n$:

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State transition with minimal energy supply

Compact control constraint set $\mathbb{U} \subset \mathbb{R}^n$, $\Phi \subset \mathbb{R}^n$ convex.

S., Philipp, Faulwasser, Worthmann, Maschke (2021). Control of port-Hamiltonian systems with minimal energy supply. EJC, 62, 33-40.

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$$\min_{u \in L_1(0,T;\mathbb{U})} \int_0^T \langle u(t), y(t) \rangle \, \mathrm{d}t$$

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- But: Non-standard OCP, as cost is linear in control.

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Optimality conditions and singular arcs

(u^*, x^*) optimal, λ Lagrange multiplier, $\mathbb{U} = [\underline{u}, \overline{u}]$

$$\begin{aligned} \dot{x}^*(t) &= \quad (J-R)Qx^*(t) + Bu^*(t) \\ \dot{\lambda}(t) &= -QBu^*(t) + Q(J+R)\lambda(t) \\ u^*(t) &\in \arg\min_{\tilde{u}\in\mathbb{U}} \; \tilde{u}^\top B^\top (Qx^*(t) + \lambda(t)) \end{aligned} \quad \rightsquigarrow$$

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Theorem (SPFWM '21)

Assume that $\operatorname{im} B \cap \ker(RQ) = \{0\}$. Then the optimal control is **completely determined** by the optimal state and the corresponding Lagrange multiplier:

$$u_{\mathcal{I}}(t) = \left(B_{\mathcal{I}}^\top QRQB_{\mathcal{I}}\right)^{-1} B_{\mathcal{I}}^\top \left[\tfrac{1}{2}(QA^2 x^*(t) + (A^2)^\top \lambda(t)) - QRQB_{\mathcal{A}} u_{\mathcal{A}}(t)\right].$$

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Subspace integral turnpike

Theorem (PSFMW '21)

Let $((J-R)Q, B)$ be controllable, $0 \in \operatorname{int} \mathbb{U}$ and assume there is a controlled steady state $(\bar{x}, \bar{u}) \in \ker R^{\frac{1}{2}} Q \times \operatorname{int} \mathbb{U}$ from which we can reach Φ.

Faulwasser, Maschke, Philipp, S., Worthmann. (2022). Optimal control of port-Hamiltonian descriptor systems with minimal energy supply, to appear in SICON

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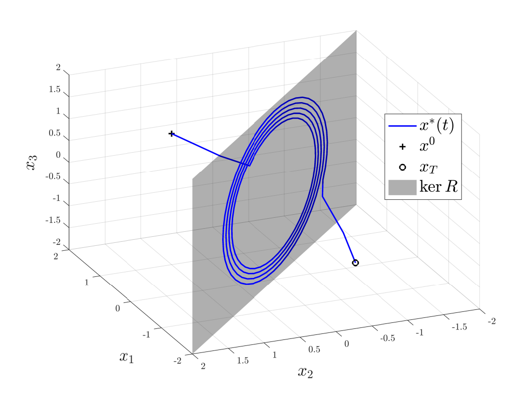
Then the pH-OCP has an **integral subspace turnpike property** with respect to $\ker R^{\frac{1}{2}} Q$ for all initial values $x_0 \in \mathbb{R}^n$, i.e, for all compact sets $K \subset \mathbb{R}^n$ there is C_K such that

$$\int_0^T \operatorname{dist}(x^*(t), \ker R^{\frac{1}{2}} Q)^2 \, dt \leq C_K.$$

Faulwasser, Maschke, Philipp, S., Worthmann. (2022). Optimal control of port-Hamiltonian descriptor systems with minimal energy supply, to appear in SICON

Solutions are close to a conservative subspace



Philipp, S., Faulwasser, Maschke, Worthmann (2021). Minimizing the energy supply of infinite-dimensional linear port-Hamiltonian systems. IFAC-PapersOnLine, 54(19), 155-160.

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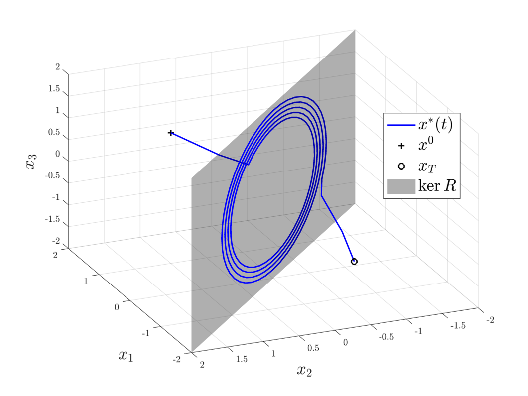
Maschke, Philipp, S., Worthmann, Faulwasser (2022). Optimal control of thermodynamic port-Hamiltonian Systems. arXiv:2202.09086.

Solutions are close to a conservative subspace



Extensions:

- Infinite dimensional systems

Philipp, S., Faulwasser, Maschke, Worthmann (2021). Minimizing the energy supply of infinite-dimensional linear port-Hamiltonian systems. IFAC-PapersOnLine, 54(19), 155-160.

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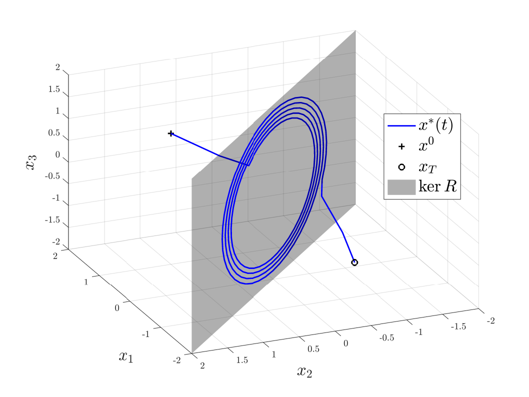

Extensions:

- Infinite dimensional systems
- DAE systems of index one

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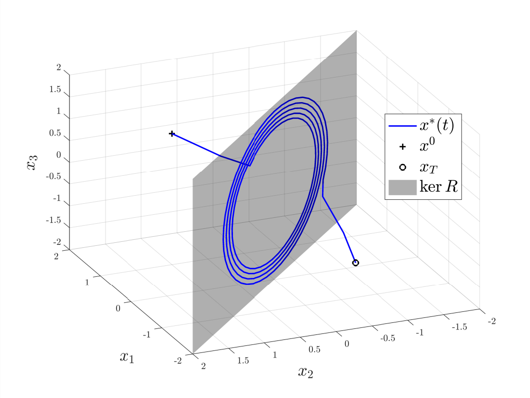

Extensions:

- Infinite dimensional systems
- DAE systems of index one
- Nonlinear thermodynamic systems:

$$\ell(x, u) = \|Cx - y\|^2 + \langle u, y \rangle$$

Philipp, S., Faulwasser, Maschke, Worthmann (2021). Minimizing the energy supply of infinite-dimensional linear port-Hamiltonian systems. IFAC-PapersOnLine, 54(19), 155-160.

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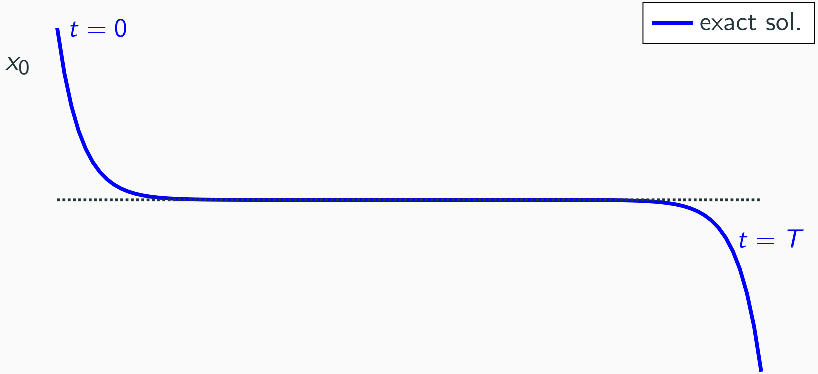
Part 4: Efficient dissipativity-exploiting numerics in optimal control

Decay of perturbations



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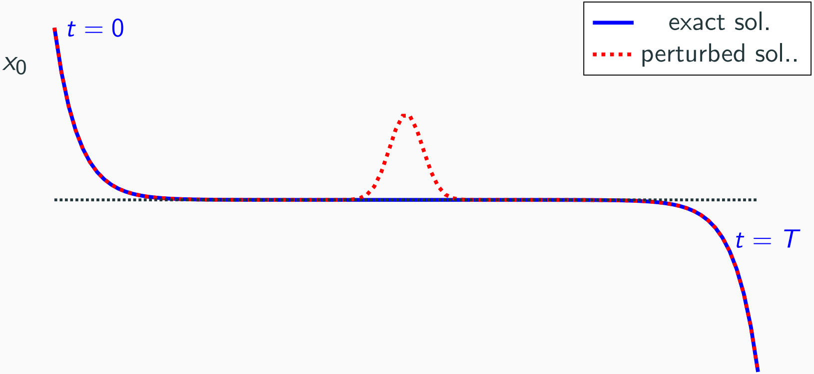
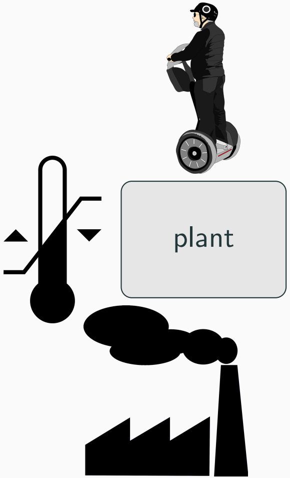
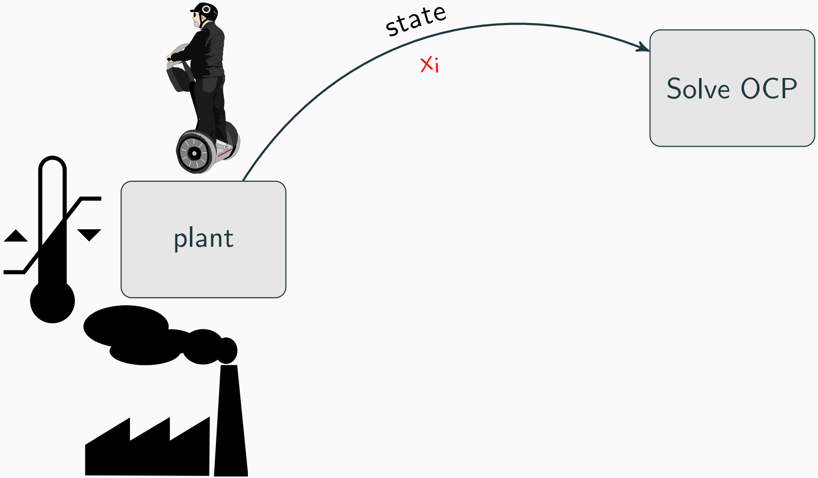


Aim: Show that perturbations stay local in time.

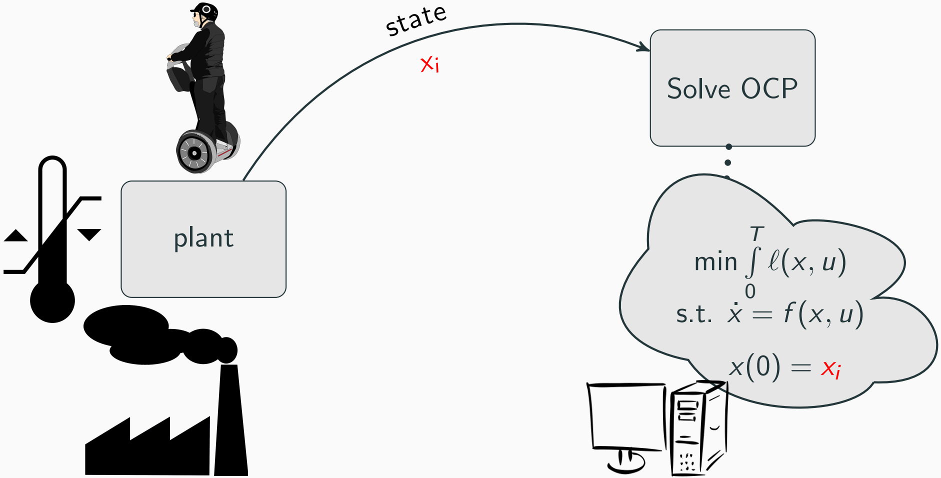
An important application: Model predictive control



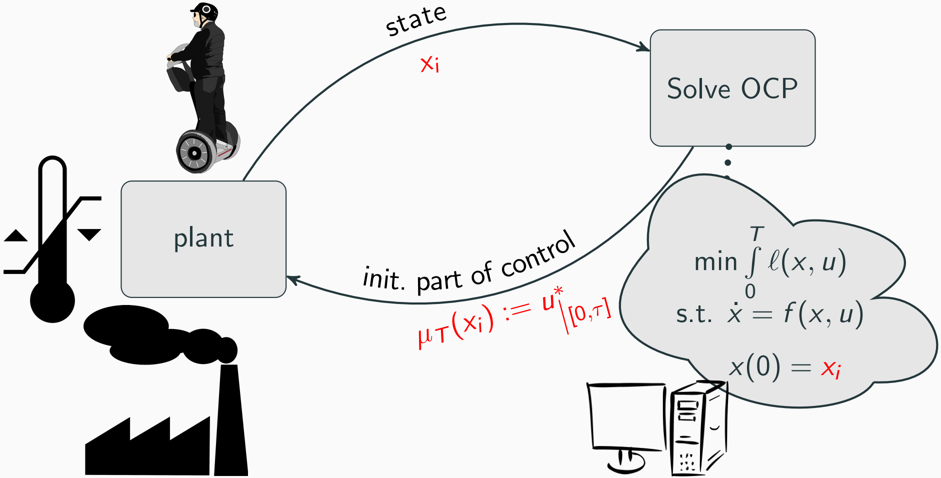
An important application: Model predictive control



An important application: Model predictive control



An important application: Model predictive control



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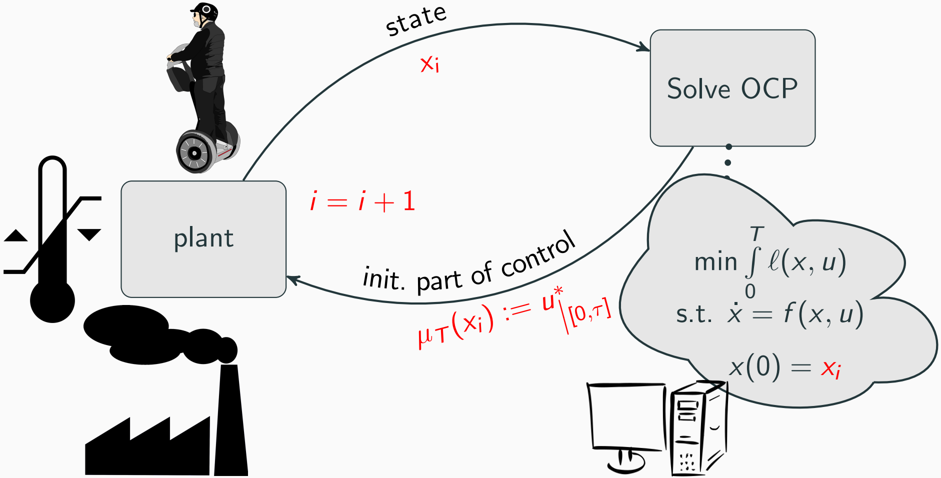
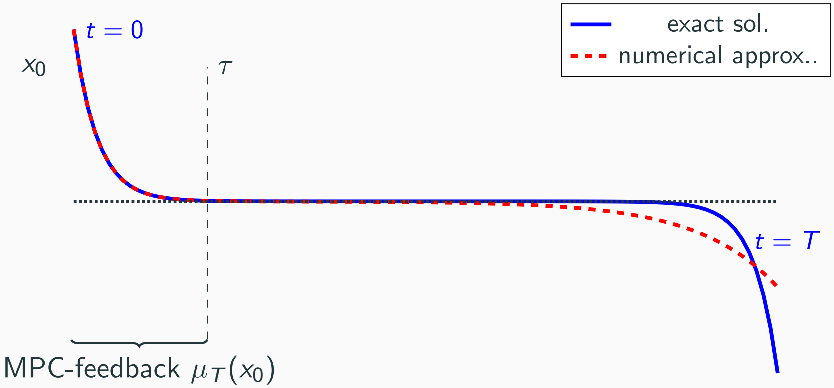


Exploiting dissipativity in numerics for MPC



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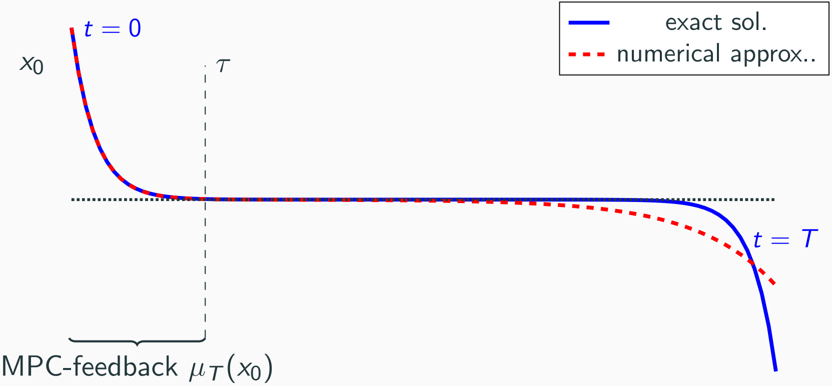

Aim: Discretization errors in the future have negligible influence on MPC-feedback.

Infinite-dimensional LQR revisited

$$\min_u \frac{1}{2}\int_0^T \|C(x(t)-x_d)\|_Y^2 + \alpha\|u(t)\|_U^2\,dt$$

$$\text{s.t.}\quad \dot{x}(t) = Ax(t) + Bu(t) + f,$$

$$x(0) = x_0$$

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$$x(0) = x_0$$

PMP

$$\begin{pmatrix}\dot{\lambda}\\ \dot{x}\end{pmatrix} = \begin{pmatrix}-A^* & -C^*C\\ -\frac{1}{\alpha}BB^* & A\end{pmatrix}\begin{pmatrix}\lambda\\ x\end{pmatrix} + \begin{pmatrix}C^*Cx_d\\ f\end{pmatrix}$$

$$x(0) = x_0,\ \lambda(T) = 0.$$

Infinite-dimensional LQR revisited

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Perturbed PMP

$$\begin{pmatrix}\dot{\tilde\lambda}\\ \dot{\tilde x}\end{pmatrix} = \begin{pmatrix}-A^* & -C^*C\\ -\frac{1}{\alpha}BB^* & A\end{pmatrix}\begin{pmatrix}\tilde\lambda\\ \tilde x\end{pmatrix} + \begin{pmatrix}C^*Cx_d\\ f\end{pmatrix} + \begin{pmatrix}\varepsilon_1\\ \varepsilon_2\end{pmatrix}$$

$$\tilde x(0) = x_0,\ \tilde\lambda(T) = 0.$$

Exponential decay of perturbations

Theorem (Grüne, S., Schiela, '19,'20)

(A, B) exp. stabilizable, (A, C) exp. detectable. Then there is $\mu, c > 0$ indep. of T, such that if

$$\|e^{-\mu\cdot}\varepsilon_{1,2}(\cdot)\|_{L_1(0,T;X)} \leq c$$

then

$$\|x(t) - \tilde{x}(t)\| + \|u(t) - \tilde{u}(t)\| + \|\lambda(t) - \tilde{\lambda}(t)\| \leq ce^{\mu t},$$

$$\|e^{-\mu\cdot}(x - \tilde{x})\|_{L_2(0,T;X)} + \|e^{-\mu\cdot}(u - \tilde{u})\|_{L_2(0,T;U)} + \|e^{-\mu\cdot}(\lambda - \tilde{\lambda})\|_{L_2(0,T;X)} \leq c.$$

Grüne, S., Schiela (2019). Sensitivity analysis of optimal control for a class of parabolic PDEs motivated by model predictive control. SICON, 57(4), 2753-2774.

Grüne, S., Schiela (2020). Exponential sensitivity and turnpike analysis for linear quadratic optimal control of general evolution equations. JDE, 268(12), 7311-7341.

Goal-oriented a posteriori refinement (Meidner '07)

Given: Quantity of interest $I(x, u)$.

Aim: Find space- and time grids, such that numerical approximation $(\tilde{x}, \tilde{u})$ has small error w.r.t. I:

$$|I(x, u) - I(\tilde{x}, \tilde{u})| < tol$$

Meidner, Vexler (2007). Adaptive space-time finite element methods for parabolic optimization problems. SICON, 46(1), 116-142.

Grüne, S., Schiela (2022). Efficient Model Predictive Control for parabolic PDEs with goal oriented error estimation. SISC, 44(1), A471-A500.

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$$I(x, u) = J(x, u) := \int_0^T \ell(x(t), u(t))\, dt.$$

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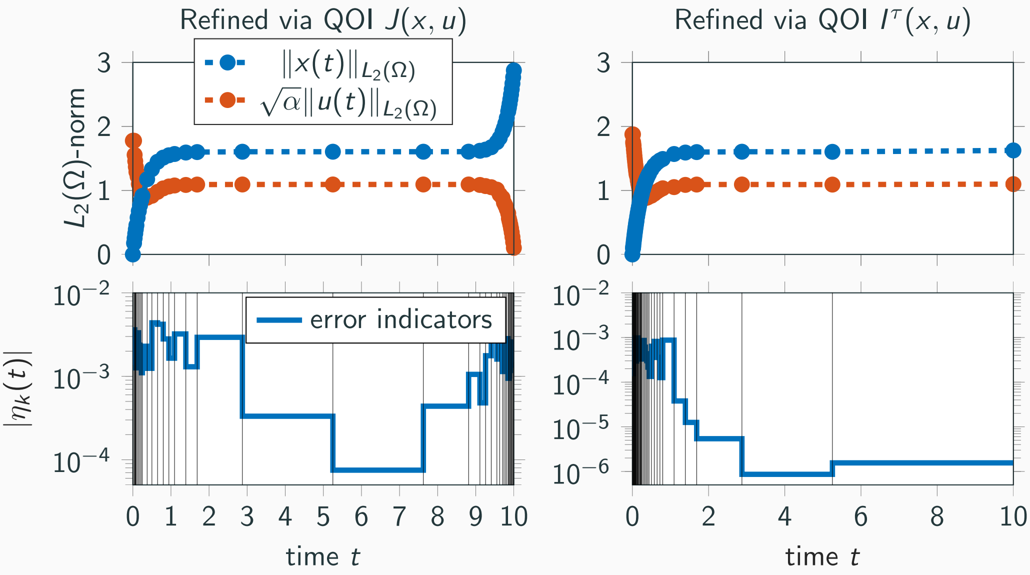
MPC-feedback is obtained by control only on $[0, \tau]$, so we may choose

$$I(x, u) = I^\tau(x, u) := \int_0^\tau \ell(x(t), u(t))\, dt.$$

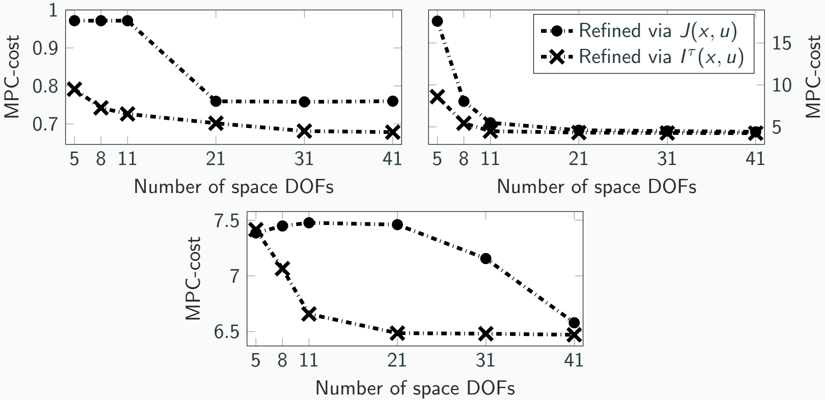
Meidner, Vexler (2007). Adaptive space-time finite element methods for parabolic optimization problems. SICON, 46(1), 116-142.

Grüne, S., Schiela (2022). Efficient Model Predictive Control for parabolic PDEs with goal oriented error estimation. SISC, 44(1), A471-A500.

Time adaptivity - grids (autonomous, unstable)



Time adaptivity - performance

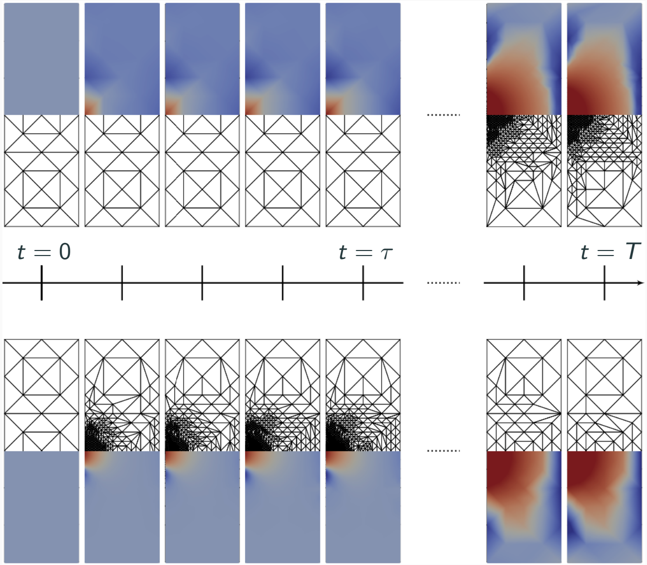


Top left: stable autonomous problem, top right: unstable autonomous problem, bottom: boundary controlled non-autonomous problem.

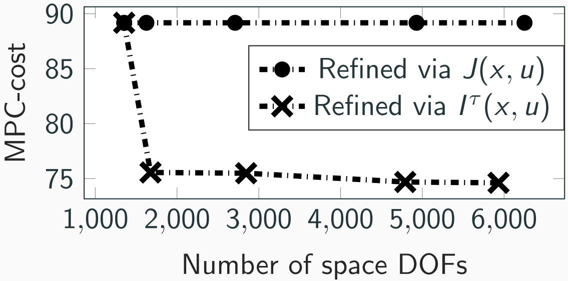
Space adaptivity - grids

QOI $J(x, u)$

QOI $I^\tau(x, u)$



Space adaptivity - performance



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Summary

- We discussed **strict dissipativity** in optimal control and **the turnpike property**.
- We considered singular dissipative optimal control of **port-Hamiltonian systems**.
- Efficient numerical methods: **stability implies locality of discretization errors**.

Summary

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Thank you.