Port-Hamiltonian Dynamic Mode Decomposition

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Fundamental Research Question

Given a quadratic **energy functional** $\mathcal{H}(x) = \frac{1}{2}x^\top Hx$, discrete **measurements** (u_i, x_i, y_i) of a system, construct a continuous dynamical system

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx + Du \end{aligned} \qquad (1)$$

as close as possible to the data in such way that any solution of (1) satisfies the **dissipation inequality**

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(x(t)) \leq y(t)^\top u(t).$$

Our Approach

Use the **port-Hamiltonian** (pH) framework and modify the **Dynamic Mode Decomposition** to be feasible for it.

The system

$$\begin{bmatrix} E\dot{x} \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}, \qquad \begin{bmatrix} A & B \\ -C & -D \end{bmatrix} = \mathcal{J} - \mathcal{R},$$

with $\mathcal{J} = \mathcal{J}^\top$ and $\mathcal{R} \succeq 0$ is called a pH system.

Remark: Port-Hamiltonian implies passivity

Key Ingredients

❶ Define discrete-time pH system

Motivated by the implicit midpoint rule we call

$$\begin{aligned} Hx_i^- &= (J - R)x_i^+ + (G - P)\widehat{u}_i, \\ \widehat{y}_i &= (G + P)^\top x_i^+ + (S - N)\widehat{u}_i, \end{aligned}$$

with

$$x_i^- := \frac{x_{i+1} - x_i}{\delta_t^i}, \quad x_i^+ := \frac{x_{i+1} + x_i}{2}, \quad \widehat{u}_i := \frac{u_{i+1} + u_i}{2}, \quad \widehat{y}_i := \frac{y_{i+1} + y_i}{2}$$

a discrete-time pH system [2].

❷ Formulate the optimization problem

Define

$$X_- := \tfrac{1}{\delta_t}\left[x_1 - x_0 \; \dots \; x_M - x_{M-1}\right], \quad X_+ := \tfrac{1}{2}\left[x_1 + x_0 \; \dots \; x_M + x_{M-1}\right],$$

$$U := \tfrac{1}{2}\left[u_1 + u_0 \; \dots \; u_M + u_{M-1}\right], \quad Y := \tfrac{1}{2}\left[y_1 + y_0 \; \dots \; y_M + y_{M-1}\right],$$

find matrices $\mathcal{J}^\star, \mathcal{R}^\star$ that solve

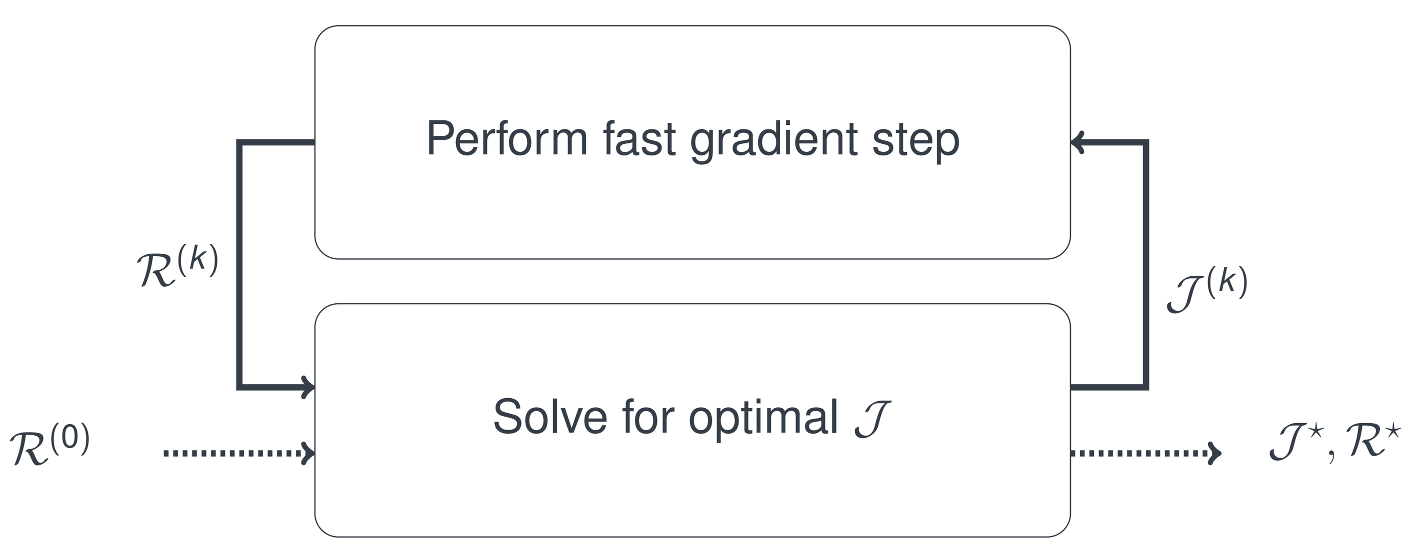
$$\min \left\| \underbrace{\begin{bmatrix} HX_- \\ -Y \end{bmatrix}}_{=:\mathcal{Z}} - (\mathcal{J} - \mathcal{R}) \underbrace{\begin{bmatrix} X_+ \\ U \end{bmatrix}}_{=:\mathcal{T}} \right\|_{\mathrm{F}} \quad \text{s.t. } \mathcal{J} = -\mathcal{J}^\top \text{ and } \widetilde{\mathcal{R}} \succeq 0. \qquad (2)$$

❸ Derive a numerical algorithm

Observation:

- For fixed $\mathcal{R}$ the optimal $\mathcal{J}$ can be computed analytically
- For fixed $\mathcal{J}$ we can employ a fast-gradient algorithm

pHDMD algorithm



❹ Initialization

Multiplying the argument of (2) with $\mathcal{T}^\top$ to obtain

$$\min \left\| \mathcal{T}^\top \mathcal{Z} - \mathcal{T}^\top(\mathcal{J} - \mathcal{R})\mathcal{T} \right\|_{\mathrm{F}} \quad \text{s.t. } \mathcal{J} = -\mathcal{J}^\top \text{ and } \mathcal{R} \succeq 0, \qquad (3)$$

which corresponds to solving the problem in a weighted semi-norm where $\mathcal{T}^\top\mathcal{T}$ is used as the semi-definite weighting matrix.

Theorem

Let $\mathcal{T}, \mathcal{Z}$ and let $V\Sigma W^\top = \mathcal{T}$ denote the skinny SVD of $\mathcal{T}$. Define

$$\widetilde{\mathcal{Z}} := \Sigma V^\top \mathcal{Z} W$$

and

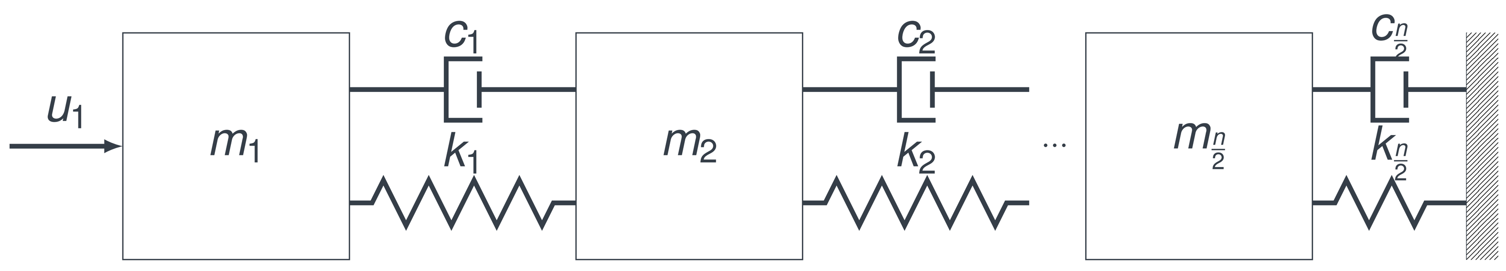
$$\mathcal{J}^\star := V\Sigma^{-1}\,\mathrm{skew}(\widetilde{\mathcal{Z}})\Sigma^{-1}V^\top \quad \text{and} \quad \mathcal{R}^\star := V\Sigma^{-1}\mathcal{P}_{\succeq}(-\widetilde{\mathcal{Z}})\Sigma^{-1}V^\top.$$

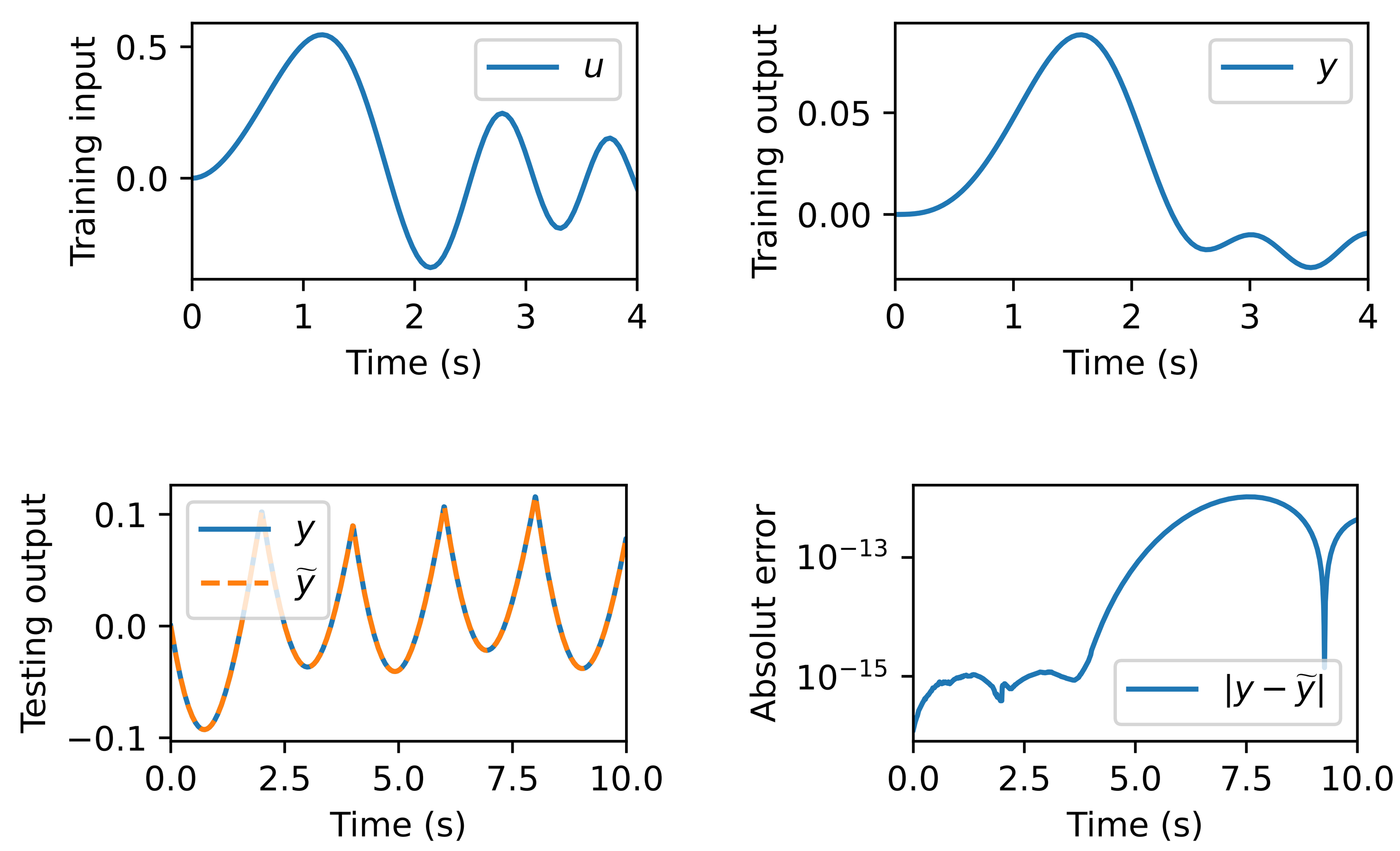
Then, $\mathcal{J}^\star$ and $\mathcal{R}^\star$ are the unique minimum-norm minimizers of (3).

Numerical Results

SISO mass-spring-damper system

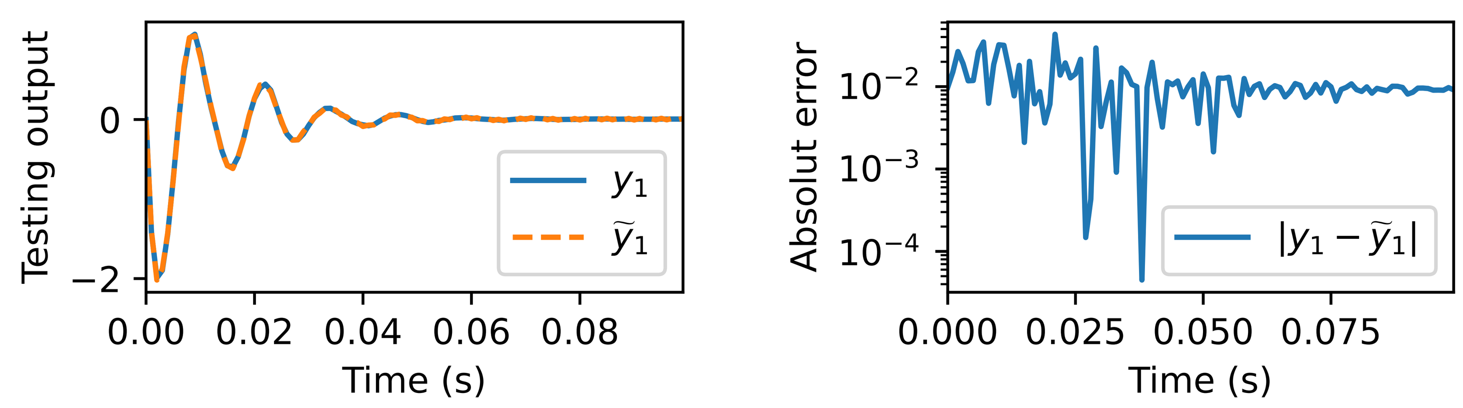
$n = 6$, $m = 1$, $\mathrm{rank}(\mathcal{T}) = 7$





MIMO poroelastic network model [1]

$n = 980$, $m = 2$, $\mathrm{rank}(\mathcal{T}) = 100 \ll 982$



References

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