Model Reduction of Port-Hamiltonian Descriptor Systems

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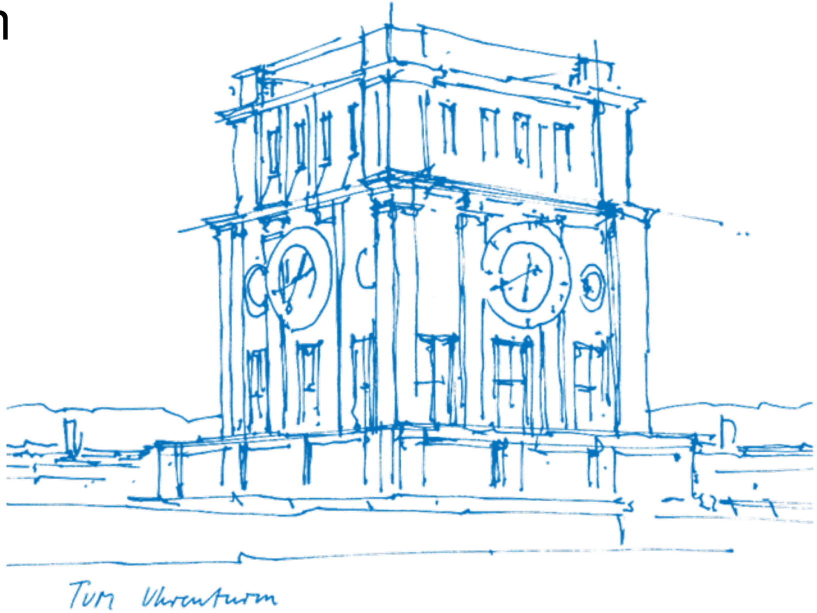
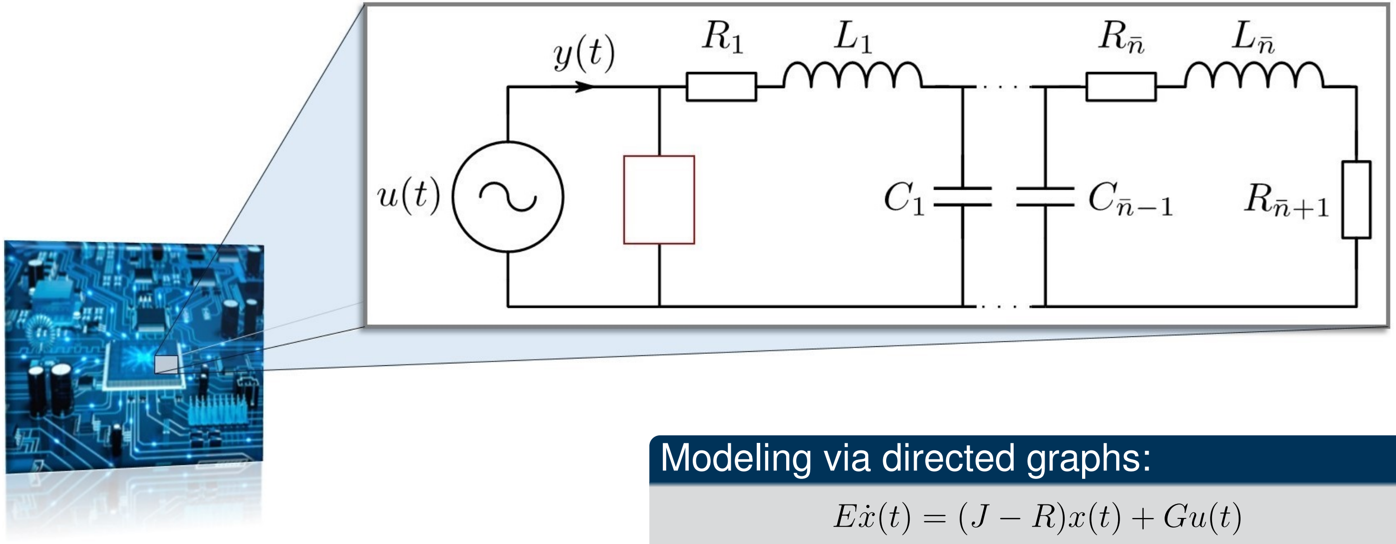


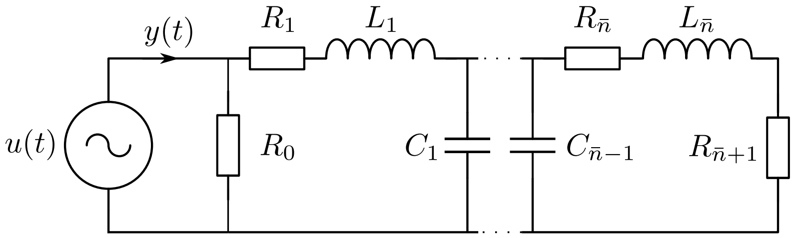
Modeling of VLSI



Modeling via directed graphs:

$$E\dot{x}(t) = (J - R)x(t) + Gu(t)$$
$$y(t) = G^{\mathsf{T}}x(t)$$

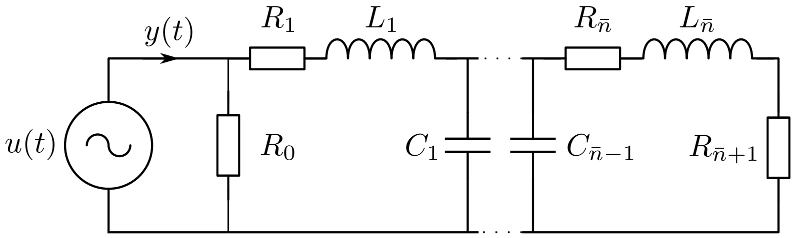
Modeling of VLSI



$$
\begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} J_{11} - R_{11} & J_{12} - R_{12} \\ J_{21} - R_{21} & J_{22} - R_{22} \end{bmatrix} x(t) + \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} u(t)
$$
$$
y = \begin{bmatrix} G_1^\mathsf{T} & G_2^\mathsf{T} \end{bmatrix} x(t)
$$

$\rightarrow$ Index-1 pHDAE

Modeling of VLSI



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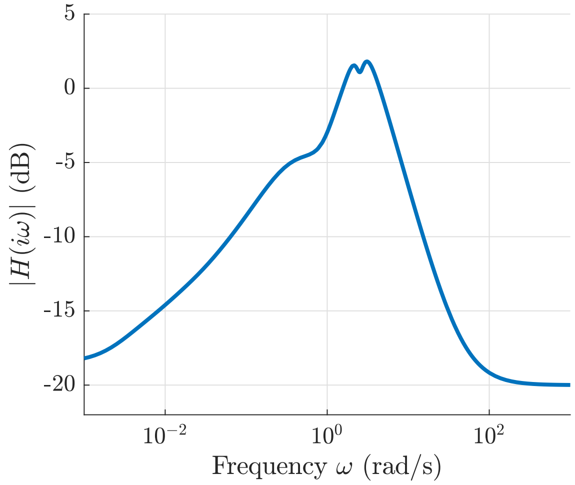
$$y = \begin{bmatrix} G_1^{\mathsf{T}} & G_2^{\mathsf{T}} \end{bmatrix} x(t)$$

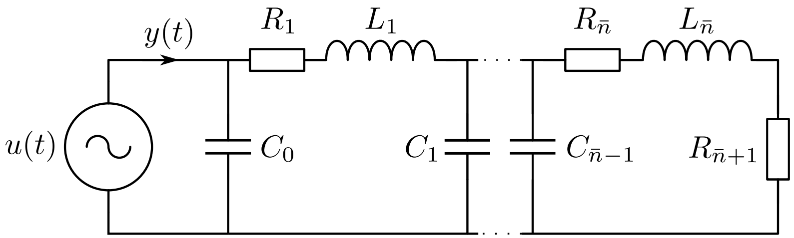
$\rightarrow$ Index-1 pHDAE

$$\begin{aligned} H(\mathrm{i}\omega) &= G^{\mathsf{T}}(\mathrm{i}\omega E - (J - R))^{-1} G \\ &= H_{\mathrm{sp}}(\mathrm{i}\omega) + \mathcal{D}_0 \end{aligned}$$

$\rightarrow$ *Proper* transfer function



Modeling of VLSI

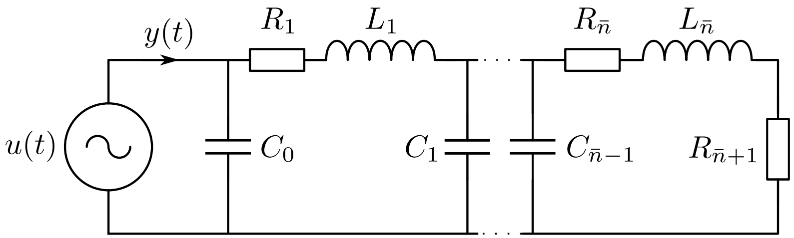


$$\begin{bmatrix} E_{11} & E_{12} & 0 & 0 \\ E_{21} & E_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \dot{x}(t) = \left(\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & 0 \\ A_{31} & A_{32} & A_{33} & 0 \\ A_{41} & 0 & 0 & 0 \end{bmatrix} \right) x(t) + \begin{bmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} G_1^\top & G_2^\top & G_3^\top & G_4^\top \end{bmatrix} x(t)$$

$\rightarrow$ Index-2 pHDAE

Modeling of VLSI



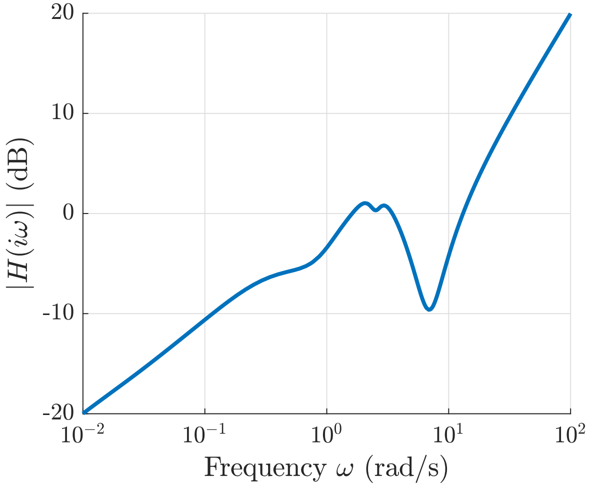
$$\begin{bmatrix} E_{11} & E_{12} & 0 & 0 \\ E_{21} & E_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \dot{x}(t) = \left(\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & 0 \\ A_{31} & A_{32} & A_{33} & 0 \\ A_{41} & 0 & 0 & 0 \end{bmatrix} \right) x(t) + \begin{bmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} G_1^\mathsf{T} & G_2^\mathsf{T} & G_3^\mathsf{T} & G_4^\mathsf{T} \end{bmatrix} x(t)$$

$\rightarrow$ Index-2 pHDAE

$$\begin{aligned} H(\mathrm{i}\omega) &= G^\mathsf{T}(\mathrm{i}\omega E - (J - R))^{-1} G \\ &= H_{\mathrm{sp}}(\mathrm{i}\omega) + \mathrm{i}\omega \cdot \mathcal{D}_1 \end{aligned}$$

$\rightarrow$ *Improper* transfer function



Approach

(1) Minimal parameterization of pHDAEs

- with any differentiation index
- with any transfer function

(2) Model order reduction (MOR)

- optimize minimal parameter sets
- obtain high-fidelity approximations
- using only samples of H

Definition of pHDAEs

Linear time-invariant differential-algebraic system

$$
\begin{aligned}
E\dot{x}(t) &= (J-R)x(t) + (G-P)u(t),\\
y(t) &= (G+P)^{\mathsf{T}}x(t) + (S-N)u(t),
\end{aligned}
$$

where $x(t) \in \mathbb{R}^n$, $u(t), y(t) \in \mathbb{R}^m$ and with

(i) *Structure matrix*

$$\Gamma := \begin{bmatrix} J & G \\ -G^{\mathsf{T}} & N \end{bmatrix} = -\Gamma^{\mathsf{T}},$$

(ii) *Dissipation matrix*

$$W := \begin{bmatrix} R & P \\ P^{\mathsf{T}} & S \end{bmatrix} = W^{\mathsf{T}} \geq 0,$$

(iii) *Hamiltonian*

$$\mathcal{H}(x(t)) := \frac{1}{2}x(t)^{\mathsf{T}}Ex(t) \geq 0$$

and transfer function

$$H(s) := (G+P)^{\mathsf{T}}(sE-(J-R))^{-1}(G-P) + S - N,$$

Properties of pHDAE transfer functions

Lemma (Wohlers, 1969)

The transfer function of every pHDAE may be decomposed into

$$H(s) = H_{\mathrm{sp}}(s) + \mathcal{D}_0 + \mathcal{D}_1 \cdot s,$$

with the following properties:

(i) The transfer function $H_{\mathrm{sp}}(s)$ is *strictly proper*, i.e. $\lim_{s\to\infty} H_{\mathrm{sp}}(s) = 0$

(ii) The *proper* subsystem with $H_{\mathrm{p}}(s) := H_{\mathrm{sp}}(s) + \mathcal{D}_0$ is *passive*

(iii) $\mathcal{D}_1 = \mathcal{D}_1^{\mathsf{T}} \geq 0$, i.e. $\mathcal{D}_1 = LL^{\mathsf{T}}$ with $L \in \mathbb{R}^{m\times q}$ and $q \leq m$

Minimal realizations of pHDAEs

Theorem

Given is a pHDAE with $x(t) \in \mathbb{R}^n$ and $H(s) = H_{\text{sp}}(s) + \mathcal{D}_0 + \mathcal{D}_1 \cdot s$.

For the *proper* transfer function $H_{\text{p}}(s) = H_{\text{sp}}(s) + \mathcal{D}_0$, there exists a realization

$$\dot{x}_{\text{p}}(t) = (J_{\text{p}} - R_{\text{p}})x_{\text{p}}(t) + (G_{\text{p}} - P_{\text{p}})u(t),$$
$$y_{\text{p}}(t) = (G_{\text{p}} + P_{\text{p}})^{\mathsf{T}} x_{\text{p}}(t) + (S_{\text{p}} - N_{\text{p}})u(t),$$

with $x_{\text{p}}(t) \in \mathbb{R}^{n_{\text{p}}}$ $(n_{\text{p}} \leq n)$.

Minimal realizations of pHDAEs

Theorem

Given is a pHDAE with $x(t) \in \mathbb{R}^n$ and $H(s) = H_{\mathrm{sp}}(s) + \mathcal{D}_0 + \mathcal{D}_1 \cdot s$.

For the *proper* transfer function $H_{\mathrm{p}}(s) = H_{\mathrm{sp}}(s) + \mathcal{D}_0$, there exists a realization

$$\begin{aligned} \dot{x}_{\mathrm{p}}(t) &= (J_{\mathrm{p}} - R_{\mathrm{p}})x_{\mathrm{p}}(t) + (G_{\mathrm{p}} - P_{\mathrm{p}})u(t), \\ y_{\mathrm{p}}(t) &= (G_{\mathrm{p}} + P_{\mathrm{p}})^{\mathsf{T}} x_{\mathrm{p}}(t) + (S_{\mathrm{p}} - N_{\mathrm{p}})u(t), \end{aligned}$$

with $x_{\mathrm{p}}(t) \in \mathbb{R}^{n_{\mathrm{p}}}$ $(n_{\mathrm{p}} \leq n)$.

Decompose $\mathcal{D}_1 = LL^{\mathsf{T}}$ with $L \in \mathbb{R}^{m \times q}$. A minimal realization of the pHDAE is given by

$$\begin{bmatrix} I_{n_{\mathrm{p}}} & 0 & 0 \\ 0 & I_q & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x}_{\min}(t) = \left(\begin{bmatrix} J_{\mathrm{p}} & 0 & 0 \\ 0 & 0 & -I_q \\ 0 & I_q & 0 \end{bmatrix} - \begin{bmatrix} R_{\mathrm{p}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) x_{\min}(t) + \left(\begin{bmatrix} G_{\mathrm{p}} \\ 0 \\ L^{\mathsf{T}} \end{bmatrix} - \begin{bmatrix} P_{\mathrm{p}} \\ 0 \\ 0 \end{bmatrix} \right) u(t),$$

$$y = \begin{bmatrix} G_{\mathrm{p}}^{\mathsf{T}} + P_{\mathrm{p}}^{\mathsf{T}} & 0 & L \end{bmatrix} x_{\min}(t) + (S_{\mathrm{p}} - N_{\mathrm{p}})u(t),$$

with $x_{\min}(t) \in \mathbb{R}^{n_{\mathrm{p}}+2q}$ and transfer function H.

Problem statement

Given: pHDAE with $x(t) \in \mathbb{R}^n$ and $H(s) = H_{\mathrm{sp}}(s) + \mathcal{D}_0 + \mathcal{D}_1 \cdot s$

Goal: Find reduced-order model (ROM)

$$\begin{bmatrix} I_r & 0 & 0 \\ 0 & I_q & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x}_r(t) = \left(\begin{bmatrix} J_{\mathrm{p},r}(\theta) & 0 & 0 \\ 0 & 0 & -I_q \\ 0 & I_q & 0 \end{bmatrix} - \begin{bmatrix} R_{\mathrm{p},r}(\theta) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) x_r(t) + \left(\begin{bmatrix} G_{\mathrm{p},r}(\theta) \\ 0 \\ L_r(\theta)^\mathsf{T} \end{bmatrix} - \begin{bmatrix} P_{\mathrm{p},r}(\theta) \\ 0 \\ 0 \end{bmatrix} \right) u(t),$$
$$y_r = \begin{bmatrix} G_{\mathrm{p},r}(\theta)^\mathsf{T} + P_{\mathrm{p},r}(\theta)^\mathsf{T} & 0 & L_r(\theta) \end{bmatrix} x_r(t) + (S_{\mathrm{p},r}(\theta) - N_{\mathrm{p},r}(\theta)) u(t),$$

with state $x_r \in \mathbb{R}^{r+2q}$ ($r \ll n$), parameterized by $\theta \in \mathbb{R}^{???}$, and transfer function $H_r(\cdot, \theta)$

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$$y_r = \begin{bmatrix} G_{\mathrm{p},r}(\theta)^{\mathsf{T}} + P_{\mathrm{p},r}(\theta)^{\mathsf{T}} & 0 & L_r(\theta) \end{bmatrix} x_r(t) + (S_{\mathrm{p},r}(\theta) - N_{\mathrm{p},r}(\theta))u(t),$$

with state $x_r \in \mathbb{R}^{r+2q}$ $(r \ll n)$, parameterized by $\theta \in \mathbb{R}^{???}$, and transfer function $H_r(\cdot, \theta)$ such that

(i) $\|H - H_r(\cdot,\theta)\|_{\mathcal{H}_2} := \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \|H(\mathrm{i}\omega) - H_r(\mathrm{i}\omega, \theta)\|_{\mathrm{F}}^2 \mathrm{d}\omega \right)^{1/2}$ is finite and *"small"*

Problem statement

Given: pHDAE with $x(t) \in \mathbb{R}^n$ and $H(s) = H_{\mathrm{sp}}(s) + \mathcal{D}_0 + \mathcal{D}_1 \cdot s$

Goal: Find reduced-order model (ROM)

$$\begin{bmatrix} I_r & 0 & 0 \\ 0 & I_q & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x}_r(t) = \left(\begin{bmatrix} J_{\mathrm{p},r}(\theta) & 0 & 0 \\ 0 & 0 & -I_q \\ 0 & I_q & 0 \end{bmatrix} - \begin{bmatrix} R_{\mathrm{p},r}(\theta) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) x_r(t) + \left(\begin{bmatrix} G_{\mathrm{p},r}(\theta) \\ 0 \\ L_r(\theta)^\mathsf{T} \end{bmatrix} - \begin{bmatrix} P_{\mathrm{p},r}(\theta) \\ 0 \\ 0 \end{bmatrix} \right) u(t),$$

$$y_r = \begin{bmatrix} G_{\mathrm{p},r}(\theta)^\mathsf{T} + P_{\mathrm{p},r}(\theta)^\mathsf{T} & 0 & L_r(\theta) \end{bmatrix} x_r(t) + (S_{\mathrm{p},r}(\theta) - N_{\mathrm{p},r}(\theta))u(t),$$

with state $x_r \in \mathbb{R}^{r+2q}$ ($r \ll n$), parameterized by $\theta \in \mathbb{R}^{???}$, and transfer function $H_r(\cdot, \theta)$ such that

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(ii) it is a pHDAE, i.e. $\Gamma_r(\theta) = \begin{bmatrix} J_{\mathrm{p},r}(\theta) & G_{\mathrm{p},r}(\theta) \\ -G_{\mathrm{p},r}(\theta)^\mathsf{T} & N_{\mathrm{p},r}(\theta) \end{bmatrix} = -\Gamma_r(\theta)^\mathsf{T}$, $\quad W_r(\theta) = \begin{bmatrix} R_{\mathrm{p},r}(\theta) & P_{\mathrm{p},r}(\theta) \\ P_{\mathrm{p},r}(\theta)^\mathsf{T} & S_{\mathrm{p},r}(\theta) \end{bmatrix} = W_r(\theta)^\mathsf{T} \geq 0$

Parameterization for $\mathcal{H}_2$ model reduction

Goal: $\|H - H_r(\cdot,\theta)\|_{\mathcal{H}_2} = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \|H(\mathrm{i}\omega) - H_r(\mathrm{i}\omega,\theta)\|_{\mathrm{F}}^2 \mathrm{d}\omega\right)^{1/2}$

$$\rightarrow\ H(\mathrm{i}\omega) - H_r(\mathrm{i}\omega,\theta) = \quad H_{\mathrm{sp}}(\mathrm{i}\omega) - H_{\mathrm{sp},r}(\mathrm{i}\omega,\theta) \quad + \quad \mathcal{D}_0 - (S_{\mathrm{p},r}(\theta) - N_{\mathrm{p},r}(\theta)) \quad + \quad \mathrm{i}\omega(\mathcal{D}_1 - L_r(\theta)L_r(\theta)^{\mathsf{T}})$$

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1. Fix L_r such that $L_r L_r^{\mathsf{T}} = \mathcal{D}_1$

Parameterization for $\mathcal{H}_2$ model reduction

Goal: $\|H - H_r(\cdot,\theta)\|_{\mathcal{H}_2} = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \|H(\mathrm{i}\omega) - H_r(\mathrm{i}\omega,\theta)\|_{\mathrm{F}}^2 \mathrm{d}\omega\right)^{1/2}$

$$\rightarrow H(\mathrm{i}\omega) - H_r(\mathrm{i}\omega,\theta) = H_{\mathrm{sp}}(\mathrm{i}\omega) - H_{\mathrm{sp},r}(\mathrm{i}\omega,\theta) + \boxed{\mathcal{D}_0 - (S_{\mathrm{p},r}(\theta) - N_{\mathrm{p},r}(\theta))} + \mathrm{i}\omega(\mathcal{D}_1 - L_r(\theta)L_r(\theta)^{\mathsf{T}})$$

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2. Fix $S_{\mathrm{p},r}, N_{\mathrm{p},r}$ such that $S_{\mathrm{p},r} - N_{\mathrm{p},r} = \mathcal{D}_0 \quad \rightarrow \quad S_{\mathrm{p},r} = \frac{1}{2}(\mathcal{D}_0^{\mathsf{T}} + \mathcal{D}_0),\; N_{\mathrm{p},r} = \frac{1}{2}(\mathcal{D}_0^{\mathsf{T}} - \mathcal{D}_0)$

Parameterization for $\mathcal{H}_2$ model reduction

Goal: $\|H - H_r(\cdot,\theta)\|_{\mathcal{H}_2} = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \|H(\mathrm{i}\omega) - H_r(\mathrm{i}\omega,\theta)\|_{\mathrm{F}}^2 \mathrm{d}\omega\right)^{1/2}$

$$\to H(\mathrm{i}\omega) - H_r(\mathrm{i}\omega,\theta) = \quad H_{\mathrm{sp}}(\mathrm{i}\omega) - H_{\mathrm{sp},r}(\mathrm{i}\omega,\theta) \quad + \quad \boxed{\mathcal{D}_0 - (S_{\mathrm{p},r}(\theta) - N_{\mathrm{p},r}(\theta))} \quad + \quad \mathrm{i}\omega(\mathcal{D}_1 - L_r(\theta)L_r(\theta)^{\mathsf{T}})$$

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AND: $\quad \Gamma_r(\theta) = \begin{bmatrix} J_{\mathrm{p},r}(\theta) & G_{\mathrm{p},r}(\theta) \\ -G_{\mathrm{p},r}(\theta)^{\mathsf{T}} & N_{\mathrm{p},r} \end{bmatrix} = -\Gamma_r(\theta)^{\mathsf{T}}, \quad W_r(\theta) = \begin{bmatrix} R_{\mathrm{p},r}(\theta) & P_{\mathrm{p},r}(\theta) \\ P_{\mathrm{p},r}(\theta)^{\mathsf{T}} & S_{\mathrm{p},r} \end{bmatrix} = W_r(\theta)^{\mathsf{T}} \geq 0$

Parameterization for $\mathcal{H}_2$ model reduction

Goal: $\|H - H_r(\cdot,\theta)\|_{\mathcal{H}_2} = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \|H(\mathrm{i}\omega) - H_r(\mathrm{i}\omega,\theta)\|_{\mathrm{F}}^2 \mathrm{d}\omega\right)^{1/2}$

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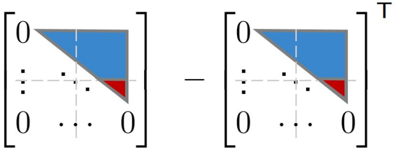
$$\Gamma_r(\theta) = \left[\begin{array}{c|c} J_{\mathrm{p},r}(\theta) & G_{\mathrm{p},r}(\theta) \\ \hline -G_{\mathrm{p},r}(\theta)^{\mathsf{T}} & N_{\mathrm{p},r} \end{array}\right] :=$$



$\rightarrow \frac{r(r+2m-1)}{2}$ parameters for θ

Parameterization for $\mathcal{H}_2$ model reduction

Goal: $\|H - H_r(\cdot,\theta)\|_{\mathcal{H}_2} = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \|H(\mathrm{i}\omega) - H_r(\mathrm{i}\omega,\theta)\|_\mathrm{F}^2 \mathrm{d}\omega\right)^{1/2}$

$$\to H(\mathrm{i}\omega) - H_r(\mathrm{i}\omega,\theta) = \quad H_\mathrm{sp}(\mathrm{i}\omega) - H_{\mathrm{sp},r}(\mathrm{i}\omega,\theta) \quad + \quad \boxed{\mathcal{D}_0 - (S_{\mathrm{p},r}(\theta) - N_{\mathrm{p},r}(\theta))} \quad + \quad \mathrm{i}\omega(\mathcal{D}_1 - L_r(\theta)L_r(\theta)^\mathsf{T})$$

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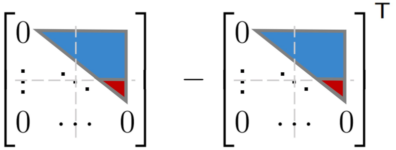
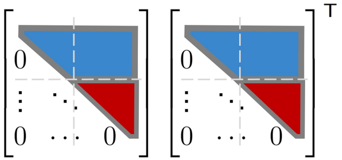
$$\Gamma_r(\theta) = \begin{bmatrix} J_{\mathrm{p},r}(\theta) & G_{\mathrm{p},r}(\theta) \\ -G_{\mathrm{p},r}(\theta)^\mathsf{T} & N_{\mathrm{p},r} \end{bmatrix} :=$$



$$\to \frac{r(r+2m-1)}{2} \text{ parameters for } \theta$$

$$W_r(\theta) = \begin{bmatrix} R_{\mathrm{p},r}(\theta) & P_{\mathrm{p},r}(\theta) \\ P_{\mathrm{p},r}(\theta)^\mathsf{T} & S_{\mathrm{p},r} \end{bmatrix} :=$$



$$\to \frac{r(r+2m+1)}{2} \text{ parameters for } \theta$$

Parameterization for $\mathcal{H}_2$ model reduction

Goal: $\|H - H_r(\cdot,\theta)\|_{\mathcal{H}_2} = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \|H(\mathrm{i}\omega) - H_r(\mathrm{i}\omega,\theta)\|_{\mathrm{F}}^2 \mathrm{d}\omega\right)^{1/2}$

$\rightarrow H(\mathrm{i}\omega) - H_r(\mathrm{i}\omega,\theta) = \quad H_{\mathrm{sp}}(\mathrm{i}\omega) - H_{\mathrm{sp},r}(\mathrm{i}\omega,\theta) \quad + \quad \mathcal{D}_0 - (S_{\mathrm{p},r}(\theta) - N_{\mathrm{p},r}(\theta)) \quad + \quad \mathrm{i}\omega(\mathcal{D}_1 - L_r(\theta)L_r(\theta)^\mathsf{T})$

1. Fix L_r such that $L_r L_r^\mathsf{T} = \mathcal{D}_1$
2. Fix $S_{\mathrm{p},r}, N_{\mathrm{p},r}$ such that $S_{\mathrm{p},r} - N_{\mathrm{p},r} = \mathcal{D}_0 \quad \rightarrow \quad S_{\mathrm{p},r} = \frac{1}{2}(\mathcal{D}_0^\mathsf{T} + \mathcal{D}_0),\ N_{\mathrm{p},r} = \frac{1}{2}(\mathcal{D}_0^\mathsf{T} - \mathcal{D}_0)$

Parameterized ROM with $\theta \in \mathbb{R}^{r(r+2m)}$

$$\begin{bmatrix} I_r & 0 & 0 \\ 0 & I_q & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x}_r(t) = \left(\begin{bmatrix} J_{\mathrm{p},r}(\theta) & 0 & 0 \\ 0 & 0 & -I_q \\ 0 & I_q & 0 \end{bmatrix} - \begin{bmatrix} R_{\mathrm{p},r}(\theta) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) x_r(t) + \left(\begin{bmatrix} G_{\mathrm{p},r}(\theta) \\ 0 \\ L_r^\mathsf{T} \end{bmatrix} - \begin{bmatrix} P_{\mathrm{p},r}(\theta) \\ 0 \\ 0 \end{bmatrix} \right) u(t)$$

$$y_r = \begin{bmatrix} G_{\mathrm{p},r}(\theta)^\mathsf{T} + P_{\mathrm{p},r}(\theta)^\mathsf{T} & 0 & L_r \end{bmatrix} x_r(t) + (S_{\mathrm{p},r} - N_{\mathrm{p},r}) u(t)$$

Optimization of θ

Given:

$$H(s) = H_{\text{sp}}(s) + \mathcal{D}_0 + \mathcal{D}_1 \cdot s$$

Assumption:

$$\begin{aligned} H_r(s,\theta) &= H_{\text{sp},r}(s,\theta) + \mathcal{D}_0 + \mathcal{D}_1 \cdot s \\ &= \sum_{i=1}^{r} \frac{l_i(\theta) r_i(\theta)^\mathsf{T}}{s - \lambda_i(\theta)} + \mathcal{D}_0 + \mathcal{D}_1 \cdot s \end{aligned}$$

Optimization of θ

Given:

$$H(s) = H_{\mathrm{sp}}(s) + \mathcal{D}_0 + \mathcal{D}_1 \cdot s$$

Assumption:

$$\begin{aligned} H_r(s,\theta) &= H_{\mathrm{sp},r}(s,\theta) + \mathcal{D}_0 + \mathcal{D}_1 \cdot s \\ &= \sum_{i=1}^{r} \frac{l_i(\theta) r_i(\theta)^{\mathsf{T}}}{s - \lambda_i(\theta)} + \mathcal{D}_0 + \mathcal{D}_1 \cdot s \end{aligned}$$

Pole-residue formulation of $\mathcal{H}_2$ error (Beattie, 2009):

$$\|H - H_r(\cdot,\theta)\|_{\mathcal{H}_2}^2 = \|H_{\mathrm{sp}}\|_{\mathcal{H}_2}^2 - 2\sum_{i=1}^{r} l_i(\theta)^{\mathsf{T}} H_{\mathrm{sp}}(-\lambda_i(\theta)) r_i(\theta) + \sum_{j,k=1}^{r} \frac{l_j(\theta)^{\mathsf{T}} l_k(\theta) r_k(\theta)^{\mathsf{T}} r_j(\theta)}{-\lambda_j(\theta) - \lambda_k(\theta)}.$$

Optimization of θ

Given:

$$H(s) = H_{\text{sp}}(s) + \mathcal{D}_0 + \mathcal{D}_1 \cdot s$$

Assumption:

$$\begin{aligned} H_r(s,\theta) &= H_{\text{sp},r}(s,\theta) + \mathcal{D}_0 + \mathcal{D}_1 \cdot s \\ &= \sum_{i=1}^{r} \frac{l_i(\theta) r_i(\theta)^{\mathsf{T}}}{s - \lambda_i(\theta)} + \mathcal{D}_0 + \mathcal{D}_1 \cdot s \end{aligned}$$

Pole-residue formulation of $\mathcal{H}_2$ error (Beattie, 2009):

$$\|H - H_r(\cdot,\theta)\|_{\mathcal{H}_2}^2 - \|H_{\text{sp}}\|_{\mathcal{H}_2}^2 = 2\sum_{i=1}^{r} l_i(\theta)^{\mathsf{T}} H_{\text{sp}}(-\lambda_i(\theta)) r_i(\theta) + \sum_{j,k=1}^{r} \frac{l_j(\theta)^{\mathsf{T}} l_k(\theta) r_k(\theta)^{\mathsf{T}} r_j(\theta)}{-\lambda_j(\theta) - \lambda_k(\theta)}.$$

Optimization of θ

Given:

$$H(s) = H_{\mathrm{sp}}(s) + \mathcal{D}_0 + \mathcal{D}_1 \cdot s$$

Assumption:

$$\begin{aligned} H_r(s,\theta) &= H_{\mathrm{sp},r}(s,\theta) + \mathcal{D}_0 + \mathcal{D}_1 \cdot s \\ &= \sum_{i=1}^{r} \frac{l_i(\theta) r_i(\theta)^{\mathsf{T}}}{s - \lambda_i(\theta)} + \mathcal{D}_0 + \mathcal{D}_1 \cdot s \end{aligned}$$

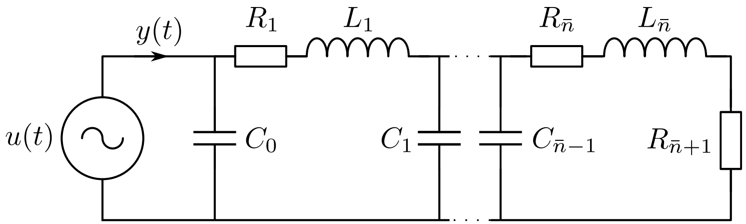
Pole-residue formulation of $\mathcal{H}_2$ error (Beattie, 2009):

$$\|H - H_r(\cdot,\theta)\|_{\mathcal{H}_2}^2 - \|H_{\mathrm{sp}}\|_{\mathcal{H}_2}^2 = 2\sum_{i=1}^{r} l_i(\theta)^{\mathsf{T}} H_{\mathrm{sp}}(-\lambda_i(\theta)) r_i(\theta) + \sum_{j,k=1}^{r} \frac{l_j(\theta)^{\mathsf{T}} l_k(\theta) r_k(\theta)^{\mathsf{T}} r_j(\theta)}{-\lambda_j(\theta) - \lambda_k(\theta)}.$$

Optimization problem

$$\min_{\theta \in \mathbb{R}^{r(r+2m)}} \mathcal{F}(\theta; H) := \|H - H_r(\cdot,\theta)\|_{\mathcal{H}_2}^2 - \|H_{\mathrm{sp}}\|_{\mathcal{H}_2}^2$$

Numerical example

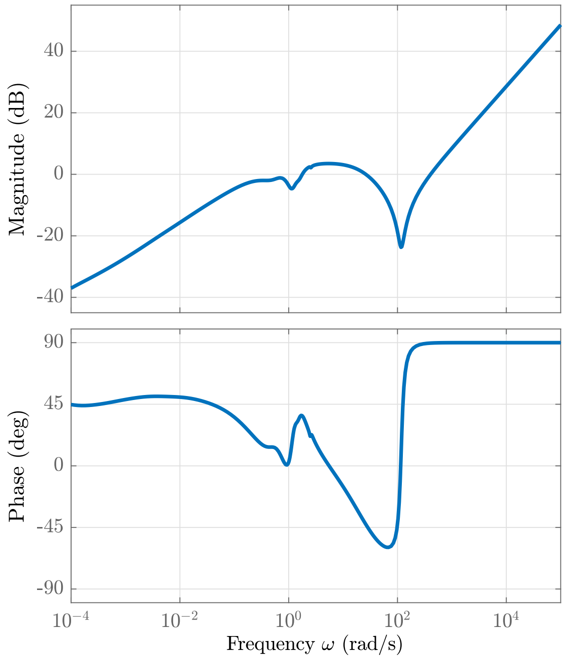


$$E\dot{x}(t) = (J - R)x(t) + Gu(t)$$
$$y(t) = G^{\mathsf{T}}x(t)$$

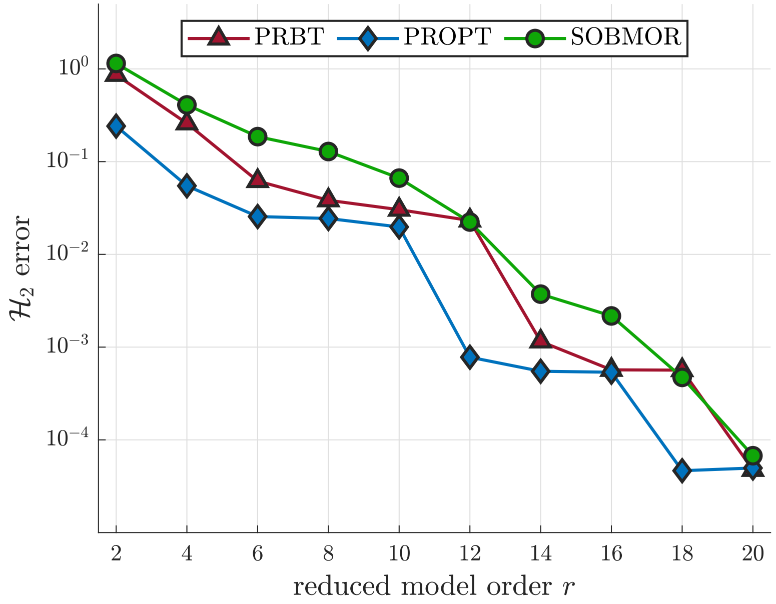
Some details:

- $\mathcal{D}_0 = 0, \quad \mathcal{D}_1 \neq 0$
- $\bar{n} = 500 \quad \rightarrow \quad x(t) \in \mathbb{R}^{1502}$
- random physical parameters $\in (0, 1]$

Bode Plot

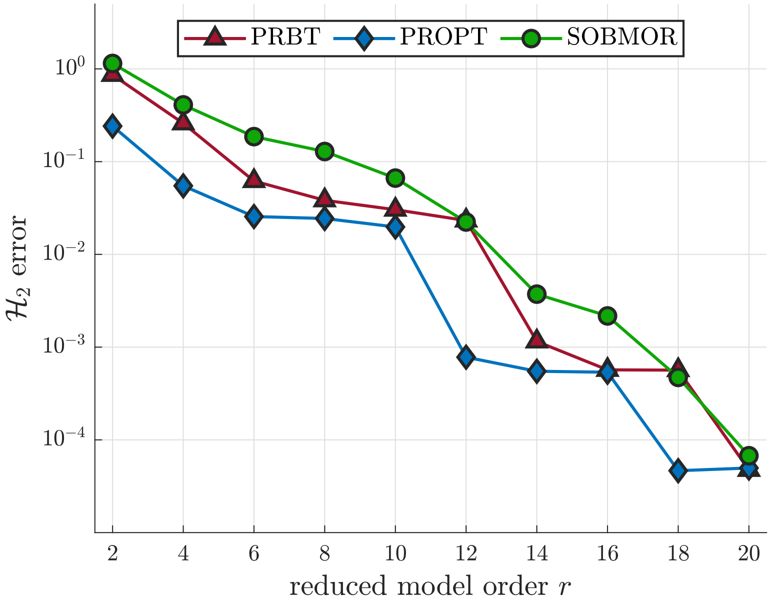


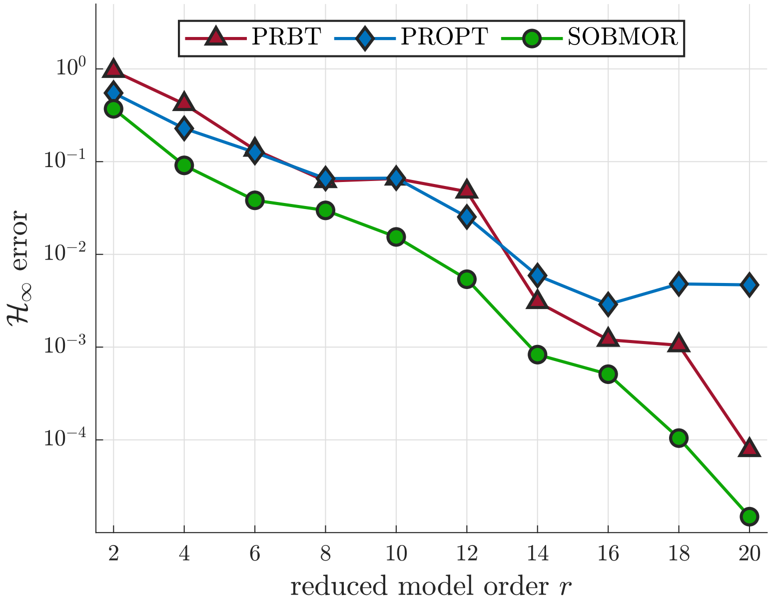
Results



PRBT
PROPT
SOBMOR
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
$\mathcal{H}_2$ error
2
4
6
8
10
12
14
16
18
20
reduced model order r

Results


PRBT
PROPT
SOBMOR
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
$\mathcal{H}_2$ error
2
4
6
8
10
12
14
16
18
20
reduced model order r


PRBT
PROPT
SOBMOR
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
$\mathcal{H}_\infty$ error
2
4
6
8
10
12
14
16
18
20
reduced model order r

Key takeaways

1. Algebraic constraints $\rightarrow$ challenging for MOR
2. pH structure $\rightarrow$ blessing and curse
3. Optimization-based approach: flexible parameterization
4. Benefits: minimal ROM dimension + only samples of H needed

MORpH - Model Reduction of pHDAEs in MATLAB

MORpH Toolbox

```
sys = phs(E,J,R,…)
```

Port-Hamiltonian

(E, J, R, Q, G, P, S, N)

```
redSys =
irkaPH(sys,r,opts)
```

```
ss2phs(),bode()
isPassive(),…
```

Non-Passive

$(E_r, A_r, B_r, C_r, D_r)$

Port-Hamiltonian

$(E_r, J_r, R_r, Q_r, G_r, P_r, S_r, N_r)$

Passive

$(E_r, A_r, B_r, C_r, D_r)$