Controllability of port-Hamiltonian systems is relative generic

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Port-Hamiltonian DAEs (pHDAEs)

Recently, it has been shown [2] that linear time-invariant DAEs

$$\boxed{\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}}$$

with $(E, A, B) \in \Sigma_{n,n,m} = \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ are **generically controllable**; for a definition see [1]. This means that a random choice of (E, A, B) yields a controllable DAE with **probability one**. We consider port-Hamiltonian DAEs

$$\boxed{\begin{array}{c}\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}\mathbf{x} = (\mathbf{J}-\mathbf{R})\mathbf{Q}\mathbf{x} + \mathbf{B}\mathbf{u}\\ J^\top = -J,\ R^\top = R \geq 0,\ E^\top Q = Q^\top E \geq 0\end{array}}$$

Within $\mathbf{V}^{\mathrm{pH}}_{\mathbf{n,n,m}} := \left\{(E, J, R, Q, B) \in (\mathbb{R}^{n\times n})^4 \times \mathbb{R}^{n\times m} \,\middle|\, J^\top = -J, R^\top = R\right\}$, denote

$$\mathbf{\Sigma}^{\mathrm{pH}}_{\mathbf{n,n,m}} := \underbrace{\left\{(E,J,R,Q,B) \in V^{\mathrm{pH}}_{n,n,m} \,\middle|\, E^\top Q = Q^\top E\right\}}_{=:\mathbb{V}} \cap \underbrace{\left\{(E,J,R,Q,B) \in V^{\mathrm{pH}}_{n,n,m} \,\middle|\, E^\top Q \geq 0\right\}}_{=:C_1} \cap \underbrace{\left\{(E,J,R,Q,B) \in V^{\mathrm{pH}}_{n,n,m} \,\middle|\, R \geq 0\right\}}_{=:C_2}$$

We show that pHDAEs are **generically controllable**, i.e.

- pHDAEs are with controllable with **probability one**
- controllability is **invariant under small perturbations**

Relative genericity

Mathematically, probability one is expressed in terms of a measure space (X, Σ, μ) as follows:

$S \subseteq X$ is **generic** $\iff$ S contains a μ**-conull set**.

Mathematically, invariance under small perturbations is expressed in terms of a topological space X as follows

$S \subseteq X$ is **generic** $\iff$ S contains an **open and dense** set.

For $\mathbb{R}^n$, Wonham uses the **Zariski topology** (see below) to unify both approaches [4]. This concept was generalized in [3] to take the additional port-Hamiltonian structure into account:

$S \subseteq \mathbb{R}^n$ is **relative generic** in a **reference set** $V \subseteq \mathbb{R}^n$ $\iff$ S contains some **Zariski open, Euclidean dense** set.

Zariski topology

The **Zariski topology** on $\mathbb{R}^n$ is

$$\mathcal{ZT} := \left\{\mathbb{R}^n \setminus \bigcap_{i=1}^{k} p_i^{-1}(\{0\}) \,\middle|\, p_i \text{ polynomial}\right\}.$$

It can be shown that

$$\mathcal{ZT} \subseteq \left\{O \subseteq \mathbb{R}^n \,\middle|\, O \text{ open conull set}\right\} \cup \{\emptyset\}$$

for the Lebesgue measure and the Euclidean topology. For **analytic submanifolds** V we have

$$\mathcal{ZT} \cap V \subseteq \left\{O \subseteq V \,\middle|\, O \text{ open conull set}\right\} \cup \{\emptyset\}$$

with the Riemann-Lebesgue measure.

Theorem on controllability

Recall the well-known algebraic criteria for five controllability concepts [1]. Denote the nonsingular points of $\mathbb{V}$ with $\operatorname{Reg}(\mathbb{V})$.
The following sets are **relative generic** in $\operatorname{Reg}(\mathbb{V}) \cap C_1 \cap C_2 \subseteq \Sigma^{\mathrm{pH}}_{n,n,m}$, i.e. have **probability one**:

$$\left\{(E,J,R,Q,B) \in \Sigma^{\mathrm{pH}}_{n,n,m} \,\middle|\, (E,(J-R)Q,B) \text{ freely initializable}\right\}$$

$$\left\{(E,J,R,Q,B) \in \Sigma^{\mathrm{pH}}_{n,n,m} \,\middle|\, (E,(J-R)Q,B) \text{ impulse controllable}\right\}$$

$$\left\{(E,J,R,Q,B) \in \Sigma^{\mathrm{pH}}_{n,n,m} \,\middle|\, (E,(J-R)Q,B) \text{ behavioural controllable}\right\}$$

$$\left\{(E,J,R,Q,B) \in \Sigma^{\mathrm{pH}}_{n,n,m} \,\middle|\, (E,(J-R)Q,B) \text{ completely controllable}\right\}$$

$$\left\{(E,J,R,Q,B) \in \Sigma^{\mathrm{pH}}_{n,n,m} \,\middle|\, (E,(J-R)Q,B) \text{ strongly controllable}\right\}$$

Open problems

- To consider the whole of $\Sigma^{\mathrm{pH}}_{n,n,m}$ as a reference set, prove that $\operatorname{Reg}(\mathbb{V})$ is Euclidean dense in $\mathbb{V}$.
- Extend our result to systems with non-quadratic coefficients, i.e. find necessary and sufficient conditions on ℓ, n, m so that $\Sigma^{\mathrm{pH}}_{\ell,n,m}$ is generically controllable.
- Prove that pHDAEs that are not ODEs are generically controllable.
  Extend our result to the reference set $\left\{(E,J,R,Q,B) \in \Sigma^{\mathrm{pH}}_{\ell,n,m} \,\middle|\, \operatorname{rk} E = r\right\}$.

References

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