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Numerical Challenges and Model Order Reduction in Gas Network Simulation

Workshop ≪Trends on dissipativity in systems and control≫

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May, 23th-25th

Dissipativity

A system[1] is given by

- Input Space $\mathcal{U} = \{u : [0, \infty) \to U\}$
- Output Space $\mathcal{Y} = \{y : [0, \infty) \to Y\}$
- State Space X
- State transition function $\phi : [0, \infty)^2 \times X \times U \to X$
- read-out function $r : X \times U \to Y$

[1] Willems, J. C. (1972). Dissipative dynamical systems—part i: general theory. Archive Rational Mechanics and Analysis, 45(5), 321–351

Dissipativity

A dissipative system[1] is given by

- Input Space $\mathcal{U} = \{u : [0, \infty) \to U\}$
- Output Space $\mathcal{Y} = \{y : [0, \infty) \to Y\}$
- State Space X
- State transition function $\phi : [0, \infty)^2 \times X \times U \to X$
- read-out function $r : X \times U \to Y$
- supply rate $w : U \times Y \to \mathbb{R}$
- storage function $S : X \to \mathbb{R}^+$ such that $S(x_T) \leq S(x_0) + \int_0^T w(t)dt$

[1] Willems, J. C. (1972). Dissipative dynamical systems—part i: general theory. Archive Rational Mechanics and Analysis, 45(5), 321–351

The Hamiltonian H

$$
\begin{aligned}
H(1) - H(0) &= \int_0^T \frac{d}{dt} H(t) dt \leq 0 \\
&= \int_0^T \frac{d}{dt} \int_0^L \mathcal{H}(\phi(x,t)) dx dt = \int_0^T \int_0^L \frac{d}{dt} \mathcal{H}(p(x,t), q(x,t)) dx dt \\
&= \int_0^T \int_0^L \mathcal{H}_p(p(t), q(t)) \dot{p} + \mathcal{H}_q(p(t), q(t)) \dot{q} dx dt \\
&= -\int_0^T \int_0^L \mathcal{H}_p(p(t), q(t)) \partial_x \mathcal{H}_q(p(t), q(t) + \mathcal{H}_q(p(t), q(t)) \partial_x \mathcal{H}_p(p(t), q(t) dx dt \\
&= -\int_0^T \int_0^L \partial_x \mathcal{H}_p(p(t), q(t)) \mathcal{H}_q(p(t), q(t) dx dt \\
&= \int_0^T \mathcal{H}_p(p(0,t), q(0,t)) \mathcal{H}_q(p(0,t), q(0,t) - \mathcal{H}_p(p(L,t), q(L,t)) \mathcal{H}_q(p(L,t), q(L,t) dt
\end{aligned}
$$

The Hamiltonian H

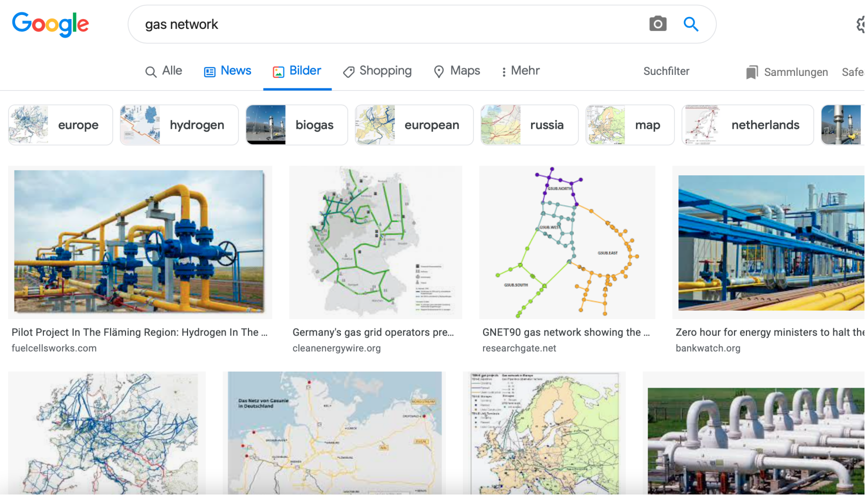
The PDE that we used in the above derivation is given by

$$\dot{p} + \partial_x \mathcal{H}_q = 0$$
$$\dot{q} + \partial_x \mathcal{H}_p = 0$$

The storage function is given by $H(t) = \int_0^L \mathcal{H}(p(x,t), q(x,t))dx$ and the supply rate is given by

$$w(t) = \mathcal{H}_p(p(0,t), q(0,t))\mathcal{H}_q(p(0,t), q(0,t) - \mathcal{H}_p(p(L,t), q(L,t))\mathcal{H}_q(p(L,t), q(L,t)$$

What is a gas network?



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Gas Transport in the Pipe

Isothermal Euler Equation

- ρ pressure density
- u velocity of the gas in the pipe
- $\rho u = q$ mass flux
- Conservation of mass and momentum

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0$$

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2 + p(\rho))}{\partial x} + g\rho h_x = -f\rho u|u|$$

- Possible boundary conditions are $\rho(0), \rho(L), u(0), u(L)$, which are input und output space candidates

Arising Questions

Which of the boundary conditions are input and which are output spaces? What is the supply rate and the storage function?

Hamiltonian or storage function

The Hamiltonian or storage function is given by

$$H(t) = \int_0^L \frac{1}{2}\rho u^2 + \rho U(\rho) dx = \int_0^L \mathcal{H}(\rho, u) dx$$

Remember the partial differential equation describing our physics

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} &= 0 \\ \frac{\partial (\rho u)}{\partial t} + \frac{\partial (\rho u^2 + p(\rho))}{\partial x} &= -f \rho u |u| \end{aligned}$$

Hamiltonian or storage function

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$$\dot{\rho} + \partial_x \mathcal{H}_u = 0$$
$$\dot{u} + \partial_x \mathcal{H}_\rho = 0$$

Hamiltonian or storage function

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$$\dot{\rho} + \partial_x \mathcal{H}_u = 0$$
$$\dot{u} + \partial_x \mathcal{H}_\rho = 0$$

Then we get a relationship between p and U.

$$2\rho U'(\rho) + \rho U''(\rho) = \frac{p'(\rho)}{\rho}$$

Equation of state

Some physically motivated description lead to nice functions U, others do not.

Boundary values as input and output

The storage function in the Hamiltonian setup

$$w(t) = \mathcal{H}_\rho(\rho(0,t), u(0,t))\mathcal{H}_u(\rho(0,t), u(0,t) - \mathcal{H}_\rho(\rho(L,t), u(L,t))\mathcal{H}_u(\rho(L,t), u_0(L,t)$$

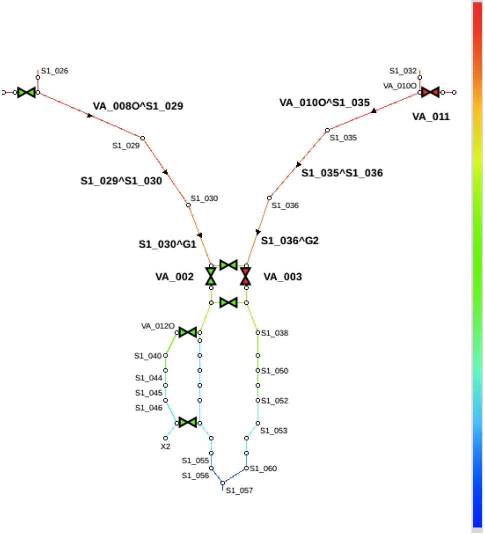
is given as a function of the boundary values of ρ and u. Remember that $\mathcal{H}(\rho, u) = \frac{1}{2}\rho u^2 + \rho U(\rho)$ and we define $u(t) = (\rho(0), q(L))$ and $y(t) = (\rho(L), q(0)$ then w is a nonlinear function of u and y.

$$\begin{aligned} w = (\frac{1}{2}u^2 + U(\rho) + \rho U'(\rho))(\rho u)\Big|_{x=L}^{x=0} &= \frac{1}{2}\rho u^3 + \rho u U(\rho) + \rho^2 u U'(\rho)\Big|_{x=L}^{x=0} \\ &= \frac{q^3}{2\rho^2} + qU(\rho) + \rho q U'(\rho)\Big|_{x=L}^{x=0} \end{aligned}$$

Friction

The friction lead to a nonequal sign in the dissipativity equation.

Gas transportation network

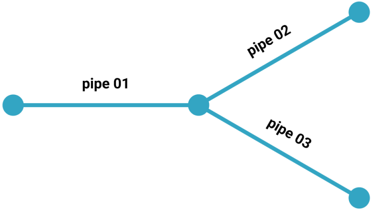


More than one pipe

- hyperbolic PDE/ system on each pipe wth input and output
- algebraic equations to connect boundary nodes correctly
- multiphase, multiple component flow

Coupling Conditions

Generalizing to a simple pipe network with 3 pipes lead to three times the same system. The Hamiltonian is then given by



$$H(t) = H_1(t) + H_2(t) + H_3(t)$$

Again define $w = \frac{d}{dt} H(t)$ as above we get

$$w = \sum_{i=1}^{3} \frac{q_i^3}{2\rho_i^2} + q_i U(\rho_i) + \rho_i q_i U'(\rho_i) \Big|_{x=L_i}^{x=0}$$

If this is independent of the interior node we want

$$\sum_{i=1}^{3} \frac{\sigma_i q_i^3}{\rho_i^2} + \sigma_i q_i U(\rho_i) + \rho_i^3 \sigma_i q_i U'(\rho_i) = 0$$

Comparing this to standard coupling conditions with $\rho_i = \rho$ and $\sum_{i=1}^{3} \sigma_i q_i = 0$ we see that this only approximately satisfy the above condition.

Gas Network System

- Overall structure is a directed graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$.
- At each node in $\mathcal{N}$ algebraic conditions are prescribed.
- The edges are the pipes described by the Euler equations.
- The resulting system looks like

$$\mathcal{M}\partial_t \phi(x,t) = \mathcal{K}\phi(x,t) + f(\phi(x,t), u(t), t)$$

  which discretized is

$$M\dot{x} = Kx + Bu + f(x,t)^2,$$

  where $\phi(x,t)$ is a vector of pressure and flux values at and $x(t)$ at different spatial points
- Depending on the network, the algebraic conditions used and the discretization schemes the matrices M, K, B and the function f can vary.
- In $u(t)$ the input functions are collected.

[2]Benner, G., Himpe, Huck,Streubel, Gas Network Benchmark Models, Springer, 2018

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Nonlinear Decoupling

Euler equation

$$\partial_t \rho(t,x) + \partial_x \rho(t,x) u(t,x) = 0,$$
$$\partial_t rho(t,x) u(t,x) + \partial_x(\rho u^2 + p(t,x)) = -\frac{f_g}{2dS^2}\rho(t,x)u(t,x)|u(t,x)|,$$

With $w^{\pm}(t,x)$ choosen correctly we get

$$\partial_t w^{\pm}(t,x) + \lambda^{\pm}\partial_x w^{\pm}(t,x) = -\frac{1}{2}\frac{f_g}{2dS^2}(\rho u)(w^+,w^-)(t,x)|u(w^+,w^-)(t,x)|.$$

- S. Grundel, M. Herty, **Hyperbolic Discretization via Riemann Invariants**

One Pipe - Speed and Accuracy

Simulation of a Pipe

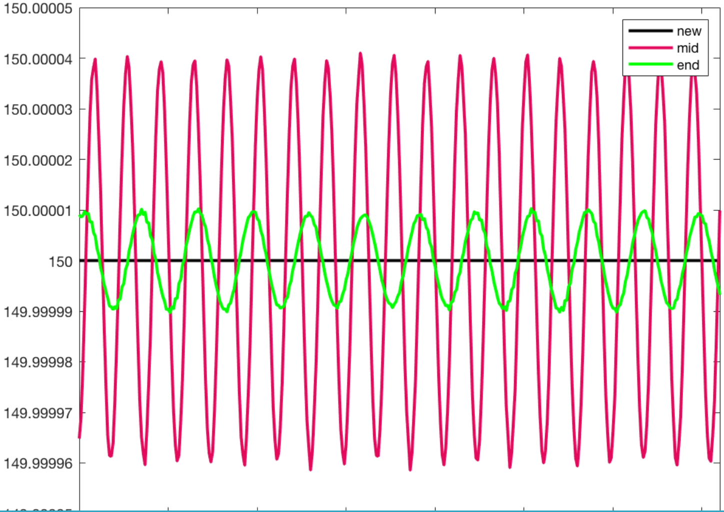
Δh	1000	300	50	10
Midpoint Discretization	68.36932	68.36932	68.36932	68.36932
Left/Right Discretization	68.36541	68.36834	68.36912	68.36928
Decoupled Discretization	68.36932	68.36932	68.36932	68.36932
True Value	68.36932	68.36932	68.36932	68.36932

Table: Accuracy of the stationary solution

Δh	250	250	100	100	10
Solver	ode15s	IMEX	ode15s	IMEX	IMEX
Midpoint Discretization	4.97	0.02	35.9	0.03	0.18
LeftRight Discretization	1.29	0.01	2.67	0.02	0.11
Decoupled Discretization	1.22	0.01	1.93	0.02	0.09

Table: Speed of a simple simulation

Oszillations of mass flux at the inlet in steady state



Well-Balancedness

Schemes that preserve steady states exactly are called well-balanced. Usually, these schemes use specific knowledge of an equilibrium state.

Theorem

The proposed scheme conserves the continuous steady state at most to order $O(\Delta x)$ and the scheme conserves discrete steady states exactly. (Provided $z(p) = c^2$ for some constant c.

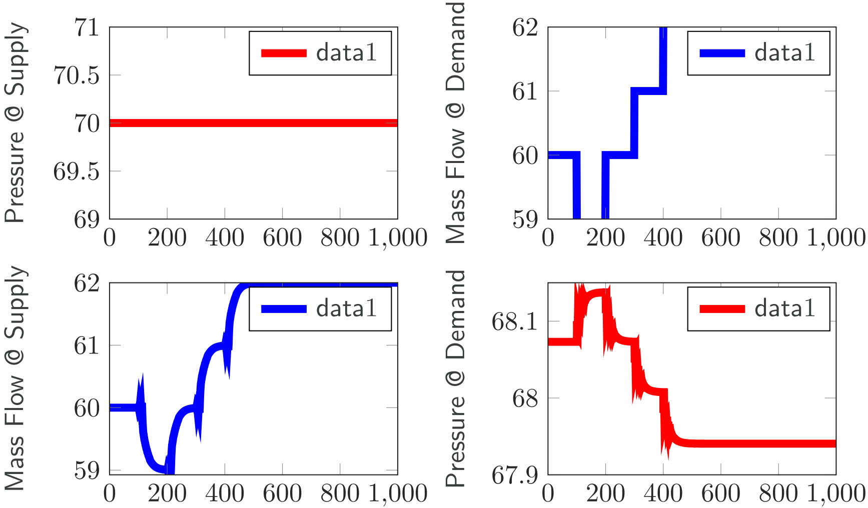
The continuous steady state is

$$\rho(x)u(x) = C_q \text{ and } \partial_x(\rho u^2 + p(x)) = \partial(C_q u(x) + p(x)) = f(\rho(x), C_q)$$

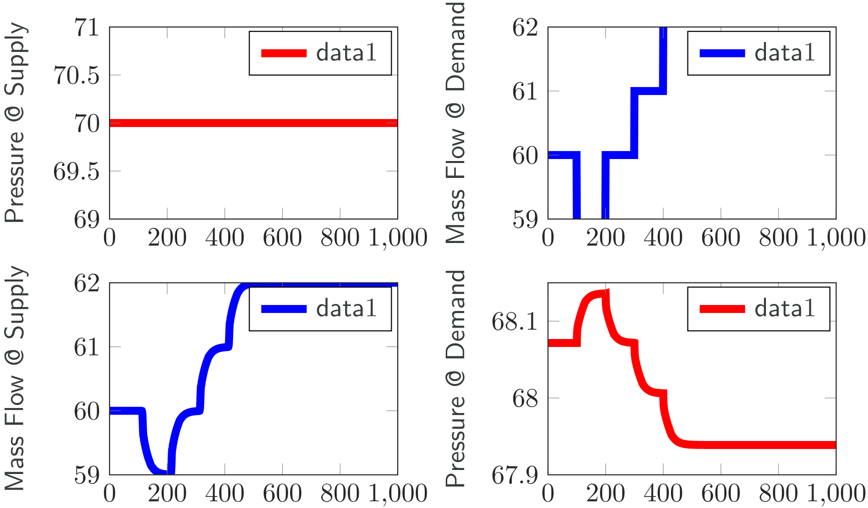
and the discretized steady states are at the cell center x_i are given by

$$C_q u(x_i) + p(x_i) = \int_0^{x_i} f(c^2 p(y), C_q) dy \text{ and } q(x_i) = C_q.$$

Dynamic Simulation Midpoint $\Delta h = 300$



Dynamic Simulation Left/Right $\Delta h = 300$



A network with cycles

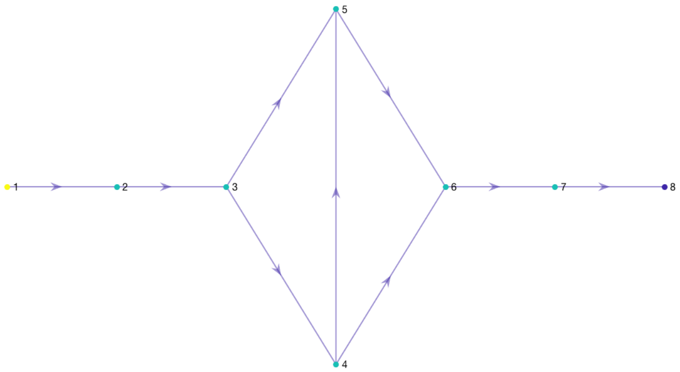
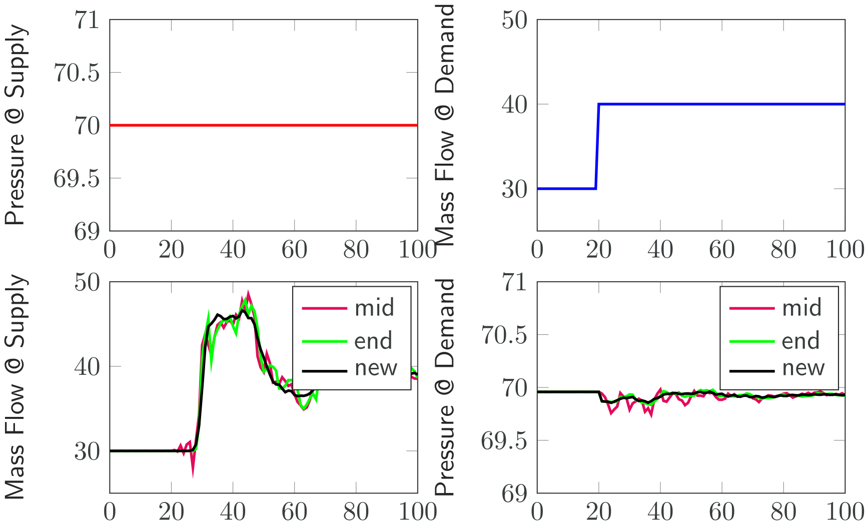


Figure: Topology of the diamond network

Numerical Simulation on the diamond network



Summary Discretization

Several ways to discretize the system to reach a Differential Algebraic Equation with different results:

- DAE properties
- Accuracy
- Well-balancedness
- Port-Hamiltonian System in discrete form

The ideal model would preserve all wanted properties and result in an easy DAE. Unforutnatley that model is not known.

DAE

All discretization methods get to a system of the form

$$M\dot{x} = Kx + Bu + f(x,t), \quad x(0) = x_0$$

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Input-Output Systems

(Possibly nonlinear) Input-Output System:

$$\begin{aligned} M\dot{x}(t) &= Kx(t) + Bu(t) + f(x(t), t), \\ y(t) &= g(x(t), u(t)) \\ x(0) &= x_0 \end{aligned}$$

- Input: $u : \mathbb{R} \to \mathbb{R}^M$
- State: $x : \mathbb{R} \to \mathbb{R}^N$
- Output: $y : \mathbb{R} \to \mathbb{R}^Q$
- Nonlinear Function: $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$
- Output Functional: $g : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}^Q$
- $N \gg 1$, $M \ll N$, $Q \ll N$

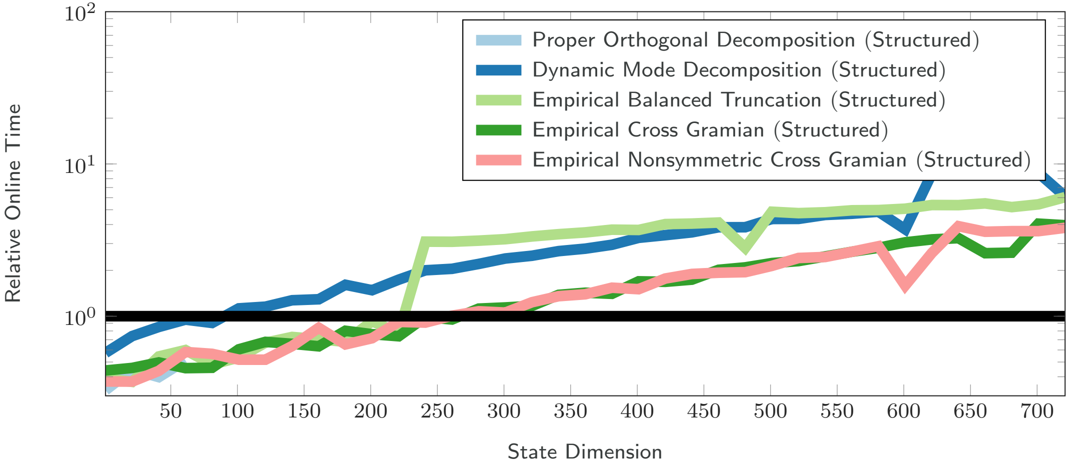
Input-to-Output Mapping:

$$u \overset{\xi}{\mapsto} x \overset{\eta}{\mapsto} y$$

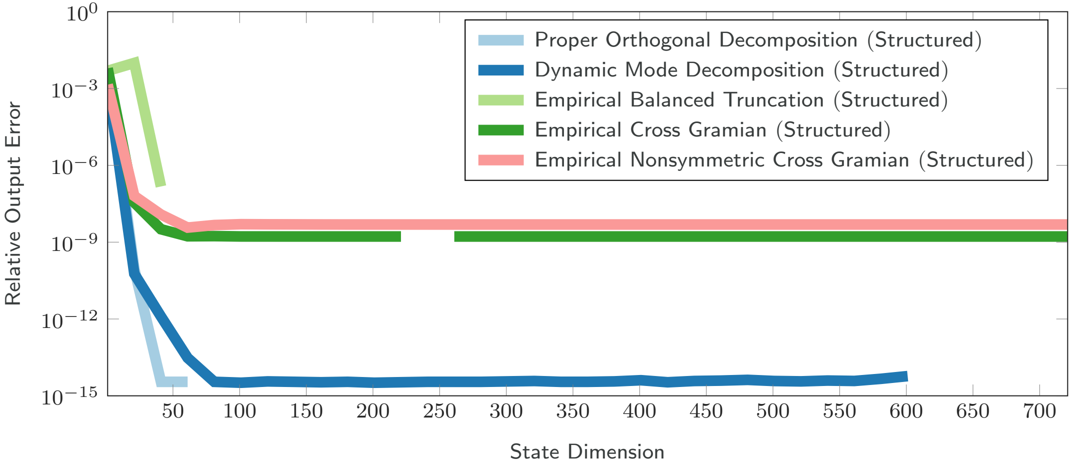
Is there a low(er) dimensional mapping $\eta \circ \xi : u \mapsto y$?

Find transformation T such that $T(x)$ is sorted by I/O importance. Then truncation is possible. (System-theoretic approach: Quantify and balance ξ and η.)

Model Order Reduction Online Time



Model Order Reduction L^∞ Error



Multi-Komponent Flow

In addition to the already discussed equation we simply add the following initial boundary value problem

$$\frac{\partial \rho Y}{\partial t} + \frac{\partial \rho u Y}{\partial x} = 0$$

which simplifies to

$$\partial_t Y + u(x,t)\partial_x Y = 0$$

This can be solved for each pipe afterwards and can be solved independently of the solution of the system before. Also reduced order modelling can be done independently.

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Basic equation

$$\frac{\partial p}{\partial t} = -\frac{1}{\gamma z S}\frac{\partial q}{\partial x} \quad \frac{\partial q}{\partial t} = -S\frac{\partial p}{\partial x} - \frac{f_g \gamma z}{2DS}\frac{q|q|}{p}$$

Decoupled equation and solution

$$w^{\pm} = \frac{1}{2}(q \pm \sqrt{\gamma z} S p) \quad \partial_t w^{\pm} \pm \frac{1}{\sqrt{\gamma z}}\partial_x w^{\pm} = \frac{1}{2} f(q,p)$$

Basic equation

$$\frac{\partial p}{\partial t} = -\frac{1}{\gamma z S}\frac{\partial q}{\partial x} \quad \frac{\partial q}{\partial t} = -S\frac{\partial p}{\partial x} - \frac{f_g \gamma z}{2DS}\frac{q|q|}{p}$$

Decoupled equation and solution

$$w^{\pm} = \frac{1}{2}(q \pm \sqrt{\gamma z} S p) \quad \partial_t w^{\pm} \pm \frac{1}{\sqrt{\gamma z}}\partial_x w^{\pm} = 0 \quad w^{\pm}(x,t) = w_0(x \mp \frac{1}{\sqrt{\gamma z}} t)$$

Basic equation

$$\frac{\partial p}{\partial t} = -\frac{1}{\gamma z S}\frac{\partial q}{\partial x} \quad \frac{\partial q}{\partial t} = -S\frac{\partial p}{\partial x} - \frac{f_g \gamma z}{2DS}\frac{q|q|}{p}$$

Decoupled equation and solution

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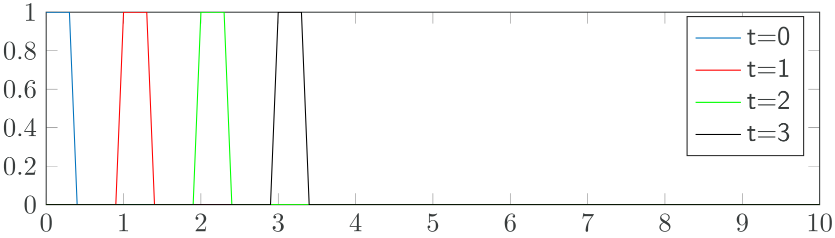

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Hyperbolic Partial Differential Equation

The Model

$$\partial_t u(\cdot,\cdot,\mu) + \nabla_x \cdot f(u(\cdot,\cdot,\mu),\mu) = 0 \text{ on } \Omega \times [0,T], \quad u(\cdot,0,\mu) = u_0(\cdot,\mu) \text{ on } \Omega$$
$$\mathcal{G}(u(\cdot,\cdot,\mu),\mu) = 0 \text{ on } \partial\Omega \times [0,T]$$

- $\mu \in \mathcal{P} \subset \mathbb{R}$ with $\mathcal{P}$ being a bounded parameter domain,
- T is the final time
- $u_0(\cdot,\mu)$ is the initial data
- $\Omega \subset \mathbb{R}^d$ is a bounded and open spatial domain
- $\mathcal{G}(\cdot,\mu)$ prescribes some boundary conditions
- $u(\cdot,t,\mu) \in \mathcal{X}$ and $u(x,t,\mu) \in \mathbb{R}$.

Solving the full order model

Finite dimensional approximation space

$$L^2(\Omega) \supset \mathcal{X}^N = \text{span}\{\phi_i, i \in \{1, \dots, N\}\}.$$

Using $\mathcal{X}^N$, we express the evolution equation for the FOM as

$$u^N(\cdot, t_{k+1}, \mu) = u^N(\cdot, t_k, \mu) + \Delta t \times L^N(u^N(\cdot, t_k, \mu), \mu), \ \forall k \in \{1, \dots, K-1\},$$

where $L^N(\cdot, \mu) : \mathcal{X}^N \to \mathcal{X}^N$ is an approximation of the original $L(\cdot, \mu)$.

Reduced dimensional approximation space

$$L^2(\Omega) \supset \mathcal{X}^m = \text{span}\{u^N(\cdot, t_k, \mu_j), j \in \{1, \dots, m\} k \in \{1, \dots, T\}\}.$$

Core of the reduction problem

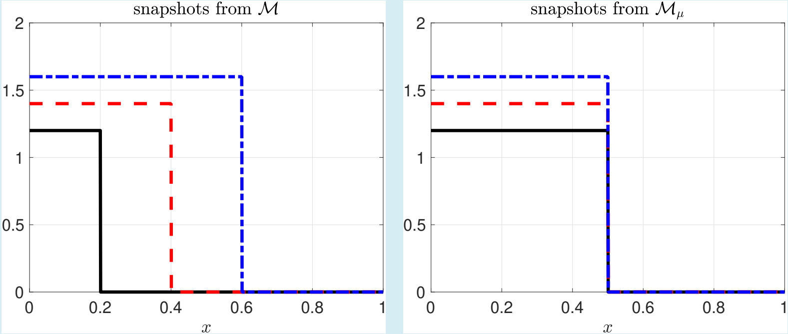


Figure: *Snapshots taken from (a) $\mathcal{M}$ and (b) $\mathcal{M}_\mu$.*

Core of the reduction problem

Consider the manifold $\mathcal{M} := \{f(\cdot,\mu) \ : \ \mu \in \mathcal{P}\} \subset L^2(\mathbb{R})$, where $f(\cdot,\mu)$ is a step function that scales and shifts to the right, and is given as

$$f(x,\mu) := \begin{cases} 1+\mu, & x \leq \mu \\ 0, & x > \mu \end{cases}, \quad \mu \in \mathcal{P} := [0,1].$$

Furthermore consider

$$\begin{aligned} \mathcal{M}_\mu := & \{f(\varphi(\cdot,\mu,\hat{\mu}),\hat{\mu}) \ : \ \varphi(\cdot,\mu,\hat{\mu}) = x - (\mu - \hat{\mu}), \ \hat{\mu} \in \mathcal{P}\}, \\ = & \{\alpha f(\cdot,\mu) \ : \ \alpha \in [1,2]\}. \end{aligned}$$

Reduced Solution manifold

For each time discretization point t_k we look for the solution in this approximation manifold;

$$\mathcal{M}_{\mu,t} := \{u^N(\varphi(\cdot,\mu,t,\mu_j,t_k),t_k,\mu_j) \; : \; j \in \{1,\dots,m\}, k \in \{1,\dots,T\}\},$$

namely

$$\mathcal{X}^n_{\mu,t_k} := \text{span}\{\psi^j_{\mu,t_k} : \psi^j_{\mu,t_k} = u^N(\varphi^M(\cdot,\mu,\hat{\mu}_j,t_k),t_k,\hat{\mu}_j),\; j \in \{1,\dots,M\}\}.$$

We compute a solution in $\mathcal{X}^n_{\mu,t_k}$ using residual-minimisation.

- Projection not possible due to nonlinearity
- Solution found through residual minimization
- Without further investigation is residual minimization expansive
- A good transform needs to be defined $\varphi(\cdot,\mu,t,\hat{t},\hat{\mu})$

Algorithms Summary

Offline Phase: Algorithm for model reduction

1. Compute the FOM for all $\mu \in \{\hat{\mu}_j\}_{j=1,\dots,M}$ using the time-evolution scheme and a finite volume scheme.
2. Compute all the snapshots of the spatial transforms $\{\varphi(x, \hat{\mu}_j, \hat{\mu}_l, t_k)\}_{j,l=1,\dots,M}$ for all $k \in \{1, \dots, K\}$.
3. Perform the offline phase of hyper-reduction.

Online Phase: Algorithm for model reduction

1. For a given μ, approximate $\{\varphi(x, \mu, \hat{\mu}_j, t_k)\}_{j=1,\dots,M}$ using polynomial interpolation.
2. Perform the online phase of hyper-reduction.
3. Compute $u^n(\cdot, t_k, \mu)$ for all $k \in \{1, \dots, K\}$ using residual-minimisation and hyper-reduction.

Example

Two dimensional Burger's equation with parameterised initial data

$$\partial_t u(\cdot,\cdot,\mu) + \frac{1}{2}\partial_x u(\cdot,\cdot,\mu)^2 + \frac{1}{2}\partial_y u(\cdot,\cdot,\mu)^2 = 0, \text{ on } \Omega \times [0,T].$$

We choose $\mathcal{P} = [1,3]$, $\Omega = [0,1]$ and $T = 0.8$. The initial data is given as

$$u_0(x,\mu) = \begin{cases} \mu \times \exp\left(-1/\left(1 - \left(\frac{\|x-\delta_1\|}{\delta_2}\right)^2\right)\right), & \frac{\|x-\delta_1\|}{\delta_2} < 1 \\ 0, & \text{else} \end{cases}.$$

We set $\delta_1 = (0.5, 0.5)^T$ and $\delta_2 = 0.2$.

- N. Sarna, S.Grundel, **Model Reduction of Time-Dependent Hyperbolic Equations using Collocated Residual Minimisation and Shifted Snapshots** submitted

We compare S-ROM (snapshots based linear ROM) and SS-ROM (shifted snapshots based non-linear ROM).

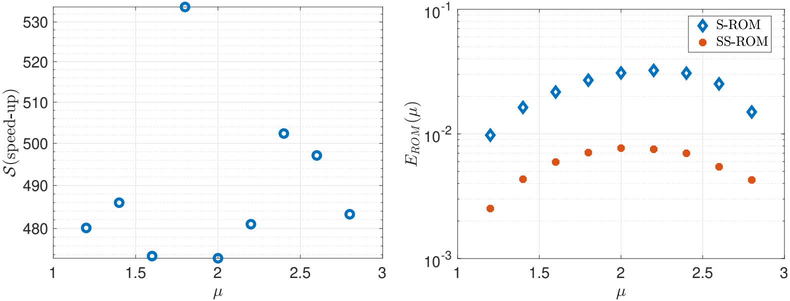


Figure: *Results for test-2. (a) Speed-up resulting from SS-ROM; and (b) E_{ROM} resulting from S-ROM and SS-ROM. Fig-(b) has a y-axis on a log-scale.*

Qualitative Results

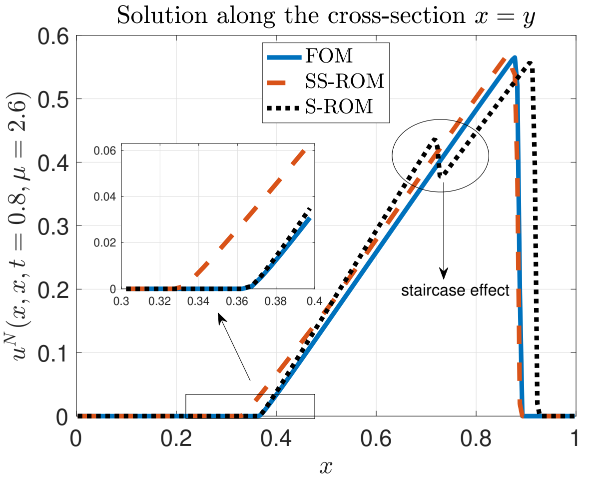
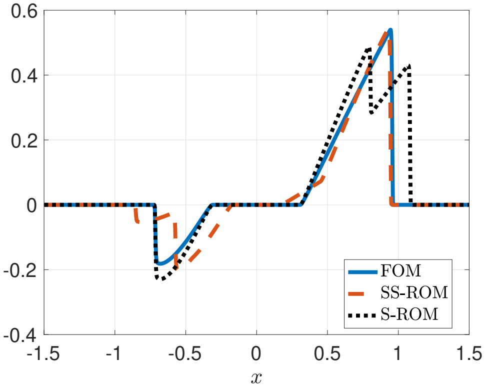


Figure: *Results for test-2. FOM and the ROM along the cross-section $x = y$ for $\mu = 2.6$ and $t = 0.8$.*

Limitations

Two Feature Initial Condition

$$u_0(x) = \begin{cases} \mu \exp\left(-1/\left(1-\left(\frac{x-\delta_1}{\delta_2}\right)^2\right)\right), & \left|\frac{x-\delta_1}{\delta_2}\right| < 1 \\ -\exp\left(-1/\left(1-\left(\frac{x+\delta_1}{\delta_2}\right)^2\right)\right), & \left|\frac{x+\delta_1}{\delta_2}\right| < 1 \\ 0, & \text{else} \end{cases}$$



Results that show the limitation of `SS-ROM`. Computed with one-dimensional Burger's equation with the initial data as given. The solutions are for $\mu = 2$ and $t = 1$.

Table of Content

Simple linear hyperbolic problem

Back to the linear hyperbolic 1-D equation:

$$\partial_t u(t,x) + \lambda \partial_x u(t,x) = g(u(t,x)),$$

where we know the spacial transformation for the zero right hand side: $u(x,t) = u_0(x - \lambda t)\}$. Based on this we can use a space time decoupled Ansatz to find a solution

$$U(t,x) = \sum_{j=1}^{N} u_j(t)\phi_j(x - \lambda t).$$

Plugging this in leads to

$$\sum_{j=1}^{N} \dot{u}_j(t)\phi_j(x-\lambda t) - \lambda \sum_{j=1}^{N} u_j(t)\phi_j'(x-\lambda t) + \lambda \sum_{j=1}^{N} u_j(t)\phi_j'(x-\lambda t) = g(u(t,x))$$

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Choosing the right test function $\Rightarrow$ ODE in all u_j

Example: Right hand side, linear, one dimensional

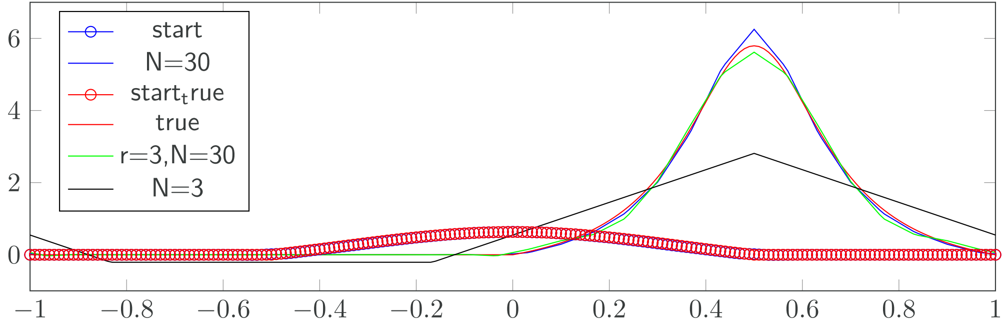

Figure: Shown is the initial condition the full order solution for $N = 30$ and the "true" solution as well as reduced solution with $r = 3$. $\partial_t w(t, x) + \lambda \partial_x w(t, x) = \gamma w^2(t, x) + \delta w$

From Nonlinear to Linear

The one dimensional hyperbolic PDE with zero right hand side

$$\partial_t u(t,x) + \partial_x f(u(t,x)) = 0$$

can be relaxed into[3]

$$\begin{aligned} \partial_t u(t,x) + \partial_x v(t,x) &= 0 \\ \partial_t v(t,x) + \lambda^2 \partial_x u(t,x) &= -\frac{1}{\epsilon}(v(t,x) - f(u(t,x))). \end{aligned}$$

Here, $\lambda > 0$ is a positive fixed parameter that fulfills the subcharacteristic condition

$$\lambda \geq \max_{x \in \mathbb{T}} |f'(u_0(x))|$$

and $\epsilon > 0$ is the (small) relaxation parameter.

[3] Chalabi, A., *Convergence of relaxation schemes for hyperbolic conservation laws with stiff source terms*, Mathematics of Computation, 1999

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Diagonalizing the system using the variables $w^{\pm}(t,x) = v(t,x) \pm \lambda\, u(t,x)$ yields the following system

$$\begin{aligned}\partial_t w^+ + \lambda \partial_x w^+ &= -\frac{1}{\epsilon}\left(\frac{w^+ + w^-}{2} - f\left(\frac{w^+ - w^-}{2\lambda}\right)\right),\\ \partial_t w^- - \lambda \partial_x w^- &= -\frac{1}{\epsilon}\left(\frac{w^+ + w^-}{2} - f\left(\frac{w^+ - w^-}{2\lambda}\right)\right).\end{aligned}$$

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Space-Time Ansatz functions

Following the procedure of the linear case we introduce $\{\phi_j(\cdot)\}_{j=1}^N$ a set of N differentiable functions $\phi_j : \mathbb{T} \to \mathbb{R}$ for $j = 1, \ldots, N$. The the solution is expanded using the translated base functions

$$w^+(t,x) \approx \sum_{j=1}^N \alpha_j^+(t)\phi_j(x - \lambda t) \text{ and } w^-(t,x) \approx \sum_{j=1}^N \alpha_j^-(t)\phi_j(x + \lambda t),$$

respectively, leading to

$$\partial_t w^+ + \lambda \partial_x w^+ = \sum_{j=1}^N \dot{\alpha}_j^+(t)\phi_j(x - \lambda t) = -\frac{1}{\epsilon}\left(\frac{w^+ + w^-}{2} - f\left(\frac{w^+ - w^-}{2\lambda}\right)\right),$$

$$\partial_t w^- - \lambda \partial_x w^- = \sum_{j=1}^N \dot{\alpha}_j^+(t)\phi_j(x + \lambda t) = -\frac{1}{\epsilon}\left(\frac{w^+ + w^-}{2} - f\left(\frac{w^+ - w^-}{2\lambda}\right)\right).$$

Differential Algebraic System

Then, the following system for the evolution of the coefficients $\alpha^{\pm} = (\alpha_j^{\pm})_{j=1}^N$ is obtained

$$M(0)\dot{\alpha}^+(t) - M(t)\dot{\alpha}^-(t) = 0$$
$$M(0)\dot{\alpha}^+(t) + M(t)\dot{\alpha}^-(t) = -\frac{2}{\epsilon}\left(\frac{1}{2}\left(M(0)\alpha^+(t) + M(t)\alpha^-(t)\right) - \tilde{F}(t,\alpha^{\pm}(t))\right),$$

where $\tilde{F} = (\tilde{F}_1, \ldots, \tilde{F}_N)$ and where

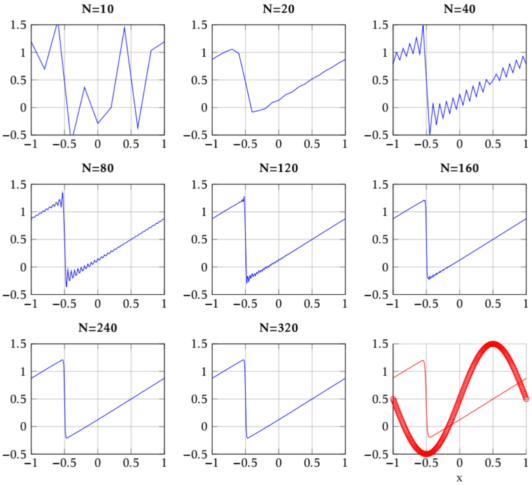
$$\tilde{F}_j(t,\alpha^{\pm}(t)) := \int_{\mathbb{T}} \phi_j(x) f(\tilde{u}(t, x+\lambda t))dx$$
$$\tilde{u}(t,x) := \frac{1}{2\lambda}\left(\sum_{j=0}^{N} \alpha_j^+(t)\phi_j(x-\lambda t) - \alpha_j^-(t)\phi_j(x+\lambda t)\right).$$

Burgers equation convergence

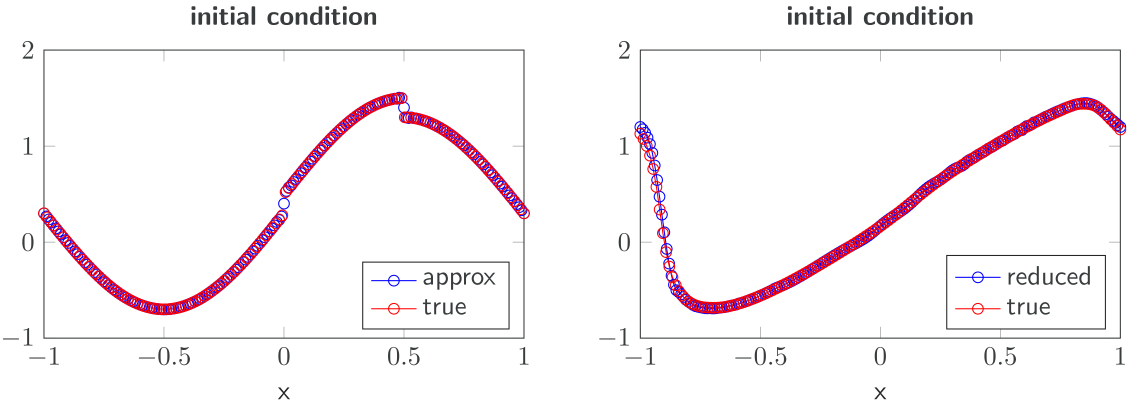
$f(u) = \frac{1}{2}u^2$. Smooth periodic initial data

$$u_0(x) = \frac{1}{2} + \sin(\pi x)$$

Relaxation approximation to a solution to Burgers equation with smooth initial data (shown in red with circles). At time $T = 1$ a shock develops that is captured by the proposed approximation for N sufficiently large (shown as continuous lines). In red a comparison with a first–order finite volume scheme with $N = 320$ discretization points in space.



Training und Testing decoupled



Left: Initial condition on the full space and reduced space with $r = 80$ out of $N = 160$ base functions.
Right: Solution at time $T = 0.3$ on full and reduced space.[4]

[4] Model-order reduction for hyperbolic relaxation systems S Grundel, M Herty International Journal of Nonlinear

Summary

- The general definition of Williams (1972) is easy and clear to work with.
- "Perfect discretization" is still work to do
- Multi component modelling can be done on top
- Hyperbolic equation pose several challenges

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Thank you for your attention