Strict Dissipativity for Multiobjective Optimal Control via Weighted Sums

Lars Grüne, Lisa Krügel, Matthias A. Müller

Mathematisches Institut, Universität Bayreuth
Institut für Regelungstechnik, Leibniz Universität Hannover

supported by DFG Deutsche Forschungsgemeinschaft

Workshop on
"Trends on dissipativity in systems and control"
Brig, Switzerland, 23-25 May 2022

Setting

Control system in discrete time

$$x_{\mathbf{u}}(k+1) = f(x_{\mathbf{u}}(k), u(k)), \quad f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n,\ n \in \mathbb{N}_0,$$

$\mathbf{u} = (u(0), u(1), \ldots)$, initial condition $x_{\mathbf{u}}(0) = x_0$

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$$J_i^N(x_0, \mathbf{u}) := \sum_{k=0}^{N-1} \ell_i(x_{\mathbf{u}}(k, x_0), u(k)) + F_i(x_{\mathbf{u}}(N, x_0)),\ i = 1, \ldots, s,$$

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Task:

$$\begin{aligned} &{}' \min_{\mathbf{u}} {}' \quad J^N(x_0, \mathbf{u}) = {}' \min_{\mathbf{u}} {}' \left(J_1^N(x_0, \mathbf{u}), \dots, J_s^N(x_0, \mathbf{u}) \right) \\ &\text{s.t.} \quad \left. \begin{aligned} & x_{\mathbf{u}}(k, x_0) \in \mathbb{X},\ k = 0, \dots, N-1, \\ & x_{\mathbf{u}}(N, x_0) \in \mathbb{X}_0, \\ & u(k) \in \mathbb{U},\ k = 0, \dots, N-1 \end{aligned} \right\} \mathbf{u} \in \mathbb{U}^N(x_0) \end{aligned}$$

Optimality in Multiobjective (MO) Optimization

What does $'\min'_{\mathbf{u}\in\mathbb{U}^N(x_0)} \big(J_1^N(x_0,\mathbf{u}),\dots,J_s^N(x_0,\mathbf{u})\big)$ mean?

$\rightsquigarrow$ Concept of optimality:

A sequence $\mathbf{u}^\star \in \mathbb{U}^N(x_0)$ is called efficient (or Pareto optimal) if there is no $\mathbf{u} \in \mathbb{U}^N(x_0)$ such that

$$\begin{aligned} \forall\, i \in \{1,\dots,s\}:\ J_i^N(x_0,\mathbf{u}) &\leq J_i^N(x_0,\mathbf{u}^\star) \text{ and} \\ \exists\, i \in \{1,\dots,s\}:\ J_i^N(x_0,\mathbf{u}) &< J_i^N(x_0,\mathbf{u}^\star) \end{aligned}$$

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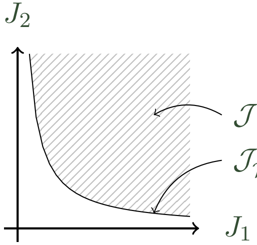

$$\mathcal{J}^N(x_0) = \{J^N(x_0,\mathbf{u}) \,|\, \mathbf{u}\in\mathbb{U}^N(x_0)\}$$

$$\mathcal{J}_{\mathcal{P}}^N(x_0) = \{J^N(x_0,\mathbf{u}) \,|\, \mathbf{u}\in\mathbb{U}_{\mathcal{P}}^N(x_0)\}$$

$\mathbb{U}_{\mathcal{P}}^N(x_0) =$ set of efficient $\mathbf{u}$

Multiobjective Model Predictive Control

The multiobjective optimisation problem "choose $\mathbf{u}^\star \in \mathbb{U}^N_{\mathcal{P}}(x)$" can be used as a building block for multiobjective MPC:

0. Choose $\mathbf{u}^\star_{x(0)} \in \mathbb{U}^N_{\mathcal{P}}(x(0))$, set $n := 0$ and go to 2.
1. Measure $x(n)$ and choose $\mathbf{u}^\star_{x(n)} \in \mathbb{U}^N_{\mathcal{P}}(x(n))$ with suitable properties
2. Apply the feedback $\mu^N(x(n)) := \mathbf{u}^\star_{x(n)}(0)$, set $n := n+1$ and go to 1.

If the "suitable properties" are properly chosen, approximate infinite horizon efficiency can be shown

[Stieler '18, Gr./Stieler '19, Eichfelder/Gr./Krügel/Schießl '22]

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$\rightsquigarrow$ search for alternative approaches

Weighted sum approach

The simplest way to solve a multiobjective optimal control problem with stage costs ℓ_i, $i = 1, \dots, s$, is to solve a standard optimal control problem with stage cost

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For non-convex efficient solution sets, at least a subset can be realised this way

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We present our results for $s = 2$ cost functions

$$\rightsquigarrow \quad \ell = \nu\ell_1 + (1-\nu)\ell_2, \quad \nu \in [0,1]$$

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Strict dissipativity: there exists a storage function $\lambda : \mathbb{X} \to \mathbb{R}$, bounded from below, and $\alpha \in \mathcal{K}_\infty$ with

$$\lambda(f(x,u)) \le \lambda(x) + \ell(x,u) - \ell(x^e,u^e) - \alpha(\|x - x^e\|)$$

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Linear quadratic problems

We start with problems with linear dynamics

$$x^+ = Ax + Bu$$

and generalised quadratic costs

$$\ell_i(x,u) = x^T Q_i x + u^T R_i u + s_i^T x + v_i^T u,$$

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Theorem: Assume that $\mathbb{Y}$ is either convex and compact or $\mathbb{Y} = \mathbb{R}^n \times \mathbb{R}^m$. Assume strict dissipativity for both ℓ_1 and ℓ_2, with optimal equilibria in the interior of $\mathbb{Y}$

Then strict dissipativity holds for $\ell_\nu = \nu\ell_1 + (1-\nu)\ell_2$ for all $\nu \in [0, 1]$

Idea of proof

Idea of proof: Strict dissipativity holds in the LQ setting if and only if it holds with linear-quadratic storage function

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This can be used to build an LQ-storage function λ_ν with

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Interestingly, while P_ν is a convex combination of P_1 and P_2, the vector p_ν in general depends nonlinearly on ν

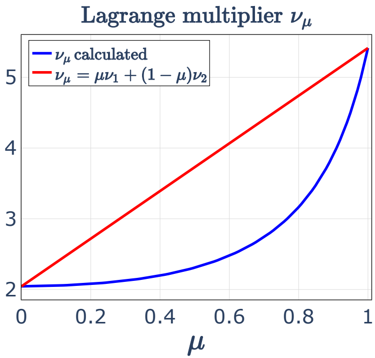
Example

Consider the 1d dynamics $x^+ = 2x + 4u$ with cost functions

$$\ell_1(x,u) = 0.1x^2 + 10u^2 + 6x + 7u$$

and

$$\ell_2(x,u) = 4x^2 + 3u^2 + 3x + 8u$$



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Note: Strict convexity of either ℓ_1 or ℓ_2 can be relaxed to mere convexity if only $\nu \in (0,1)$ is considered

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For LQ problems, in this case the ansatz

$$\lambda_\nu = \nu\lambda_1 + (1-\nu)\lambda_2 + \text{"linear correction"}$$

was successful.

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Idea of proof: In this particular case one checks that $\lambda_\nu = \nu\lambda_1 + (1-\nu)\lambda_2$ is a storage function for ℓ_ν

However, if the optimal equilibrium (x^e_ν, u^e_ν) "moves" with ν, then this will not work

For LQ problems, in this case the ansatz

$$\lambda_\nu = \nu\lambda_1 + (1-\nu)\lambda_2 + \text{"linear correction"}$$

was successful. Does this work for nonlinear problems?

Linear correction

It is known from [Faulwasser/Zanon '18] that under our standard assumptions the derivative of the storage function in the equilibrium satisfies

$$D\lambda(x^e) = p,$$

where p is (as before) the Lagrange multiplier of the problem

$$\min \ell_\nu(x,u) \quad \text{s.t.} \quad f(x,u) = x,\ (x,u) \in \mathbb{Y}$$

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In other words: In order to check whether this ansatz yields a valid storage function, we only need to check one linear correction, not many of them

Linear correction — example

$$x^+ = f(x,u) = 2x - x^2 + u + u^2 + u^3$$

with cost functions

$$\ell_1(x,u) = 2x^2 + 0.0001u^2 \quad \text{and} \quad \ell_2(x,u) = 2x^2 + 0.9999u^2 + 2u.$$

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However, for this storage function one checks that dissipativity is violated at $x = x^e$

Fully nonlinear problems — sufficient conditions

Theorem: Under suitable uniform lower bounds on the second derivatives of

$$\tilde{\ell}_\nu(x,u) = \nu\tilde{\ell}_1(x,u) + (1-\nu)\tilde{\ell}_2(x,u),$$

in $(x,u) = (x^e_\nu, u^e_\nu)$ with rotated costs

$$\tilde{\ell}_i(x,u) := \ell_i(x,u) - \ell_i(x^e_i, u^e_i) + \lambda_i(x) - \lambda_i(f(x,u))$$

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for the linear corrections $\tilde{p}_\nu$, strict dissipativity holds for $\ell_\nu = \nu\ell_1 + (1-\nu)\ell_2$ for all $\nu \in [0,1]$

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Idea of proof: Use KKT conditions

Example

For $x^+ = x^3 - 2x^2 + u$ and the two stage costs

$$\begin{aligned} \ell_1(x,u) &= \ln(5x^{0.34} - u), \\ \ell_2(x,u) &= \ln(3x^{0.2} - u), \end{aligned}$$

we can show that the lower bounds requested in the theorem hold

$\Rightarrow$ strict dissipativity holds for all weights $\nu \in [0,1]$

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Can we also find situations where we can prove that strict dissipativity is lost?

Fully nonlinear problems — necessary condition

Theorem: Assume strict dissipativity for the cost function $\ell_\nu = \nu\ell_1 + (1-\nu)\ell_2$ for all $\nu \in [\underline{\nu}, \overline{\nu}] \subseteq [0, 1]$ and that the corresponding optimal equilibria (x^e_ν, u^e_ν) are contained in a compact set $\widehat{\mathbb{Y}} \subset \mathbb{Y}$. Then the map

$$\nu \mapsto x^e_\nu$$

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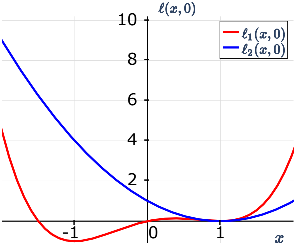
Idea of proof: Strict dissipativity implies the existence of a strictly globally optimal equilibrium $x^{e,}$. However, at any point of discontinuity there are two different optimal equilibria with identical objective value. Hence, x^e cannot exist

Example

Consider the dynamics $x^+ = x + u$ and the cost functions

$$\ell_1(x,u) = \frac{1}{2}x^4 - \frac{1}{4}x^3 - x^2 + \frac{3}{4}x + u^2 \quad \text{and} \quad \ell_2(x,u) = (x-1)^2 + u^2$$



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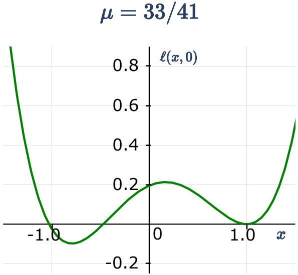

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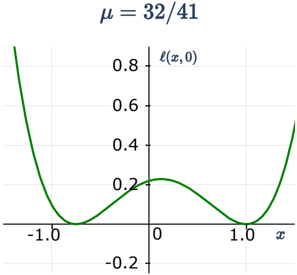

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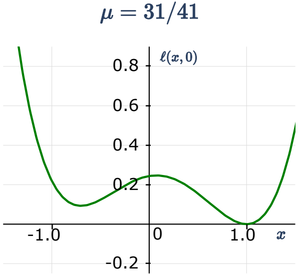

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Conclusion

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- For linear dynamics and linear-quadratic or strictly convex costs strict dissipativity persists for all weights, but the corresponding storage functions are more complicated than expected
- For nonlinear systems, the problem is surprisingly complicated. We could find a couple of interesting insights but not a simple condition that would apply to a large class of systems